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On Isometric Immersions of the Lobachevsky Plane into 4-Dimensional Euclidean Space with Flat Normal Connection

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According to Hilbert's theorem, the Lobachevsky plane L^2 does not admit a regular isometric immersion into E^3. The question on the existence of isometric immersion of L^2 into E^4 remains open. We consider isometric immersions into E^4 with flat normal connection and find a fundamental system of two partial differential equations of the second order for two functions. We prove the theorems on the non-existence of global and local isometric immersions for the case under consideration.

Key words: isometric immersion, indicatrix, curvature, asymptotic line

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1. Introduction

Hypothesis: *There exists no isometric immersion with flat normal connection of a whole Lobachevsky plane L^2 into 4-dimensional Euclidean space E^4.*

We prove the following theorem.

Theorem A. *If $F^2 \subset E^4$ is a C^3-regular immersed surface with flat normal connection isometric to the Lobachevsky plane L^2, then the metric of F^2 admits a conformal Chebyshev parametrization*

$$ds^2 = \frac{dl^2}{\sqrt{1+\beta^2}}, \quad dl^2 = dp^2 + 2\cos\omega\, dp\, dq + dq^2.$$

There is no regular isometric immersion with flat normal connection of L^2 into E^4 under which the curvature of the metric dl^2 does not change the sign or changes the sign at a finite number of bounded domains.

We remark that the functions β and ω have a geometrical meaning. The function $\beta(x)$ is equal up to a sign to the distance from $x \in F^2$ to the segment of indicatrix of normal curvature, and $\omega(x)$ is the angle between asymptotic lines with respect to normal vector that is parallel to the segment of normal curvature.

Note some results on isometric immersions of Lobachevsky plane into the Euclidean space.

E. Rozendorn constructed an isometric immersion of L^2 into E^5 in [5].

D. Bolotov proved the following theorem in [1].

Theorem B. *Denote by H the mean curvature vector. The Lobachevsky space L^n does not admit a regular isometric immersion into the Euclidean space E^{n+m} such that $|H| < const$ and the normal connection of the immersion is flat.*

On the other hand, an arbitrary geodesic disk on L^2 admits an isometric immersion into E^3.

Note also recent interesting papers [2] and [4].

2. Proof of Theorem A

Proof of Theorem A. First, we consider a local isometric immersion or immersion of a bounded domain.

Suppose that there exists an isometric immersion of a domain on the Lobachevsky plane L^2 into E^4 as a regular surface F^2. If the normal connection of F^2 is flat, then the ellipse of normal curvature degenerates into a segment γ. Denote by n_1, n_2 the unit normal frame on F^2 such that n_1 is parallel to γ and n_2 is orthogonal to γ. Denote by (u, v) the local coordinates on F^2. Let τ_1, τ_2 be unit vectors tangent to (u, v) coordinate lines, respectively. Let the end of the normal curvature vector $k_1(\tau_1)$ at $x \in F^2$ coincide with the end of γ. In the normal plane of F^2, introduce the orthogonal coordinate system (α, β) using n_1, n_2 as its basis. Denote by a a half of the length of γ. The Gauss curvature K of F^2 can be expressed [3] as follows:

$$K = \alpha^2 + \beta^2 - a^2.$$

Write the metric of F^2 as

$$ds^2 = E\,du^2 + G\,dv^2,$$

and the second quadratic forms as

$$II^\sigma = L^\sigma_{ij}\,du^i du^j, \quad \sigma = 1, 2,$$

where $u^1 = u$, $u^2 = v$. Due to the choice of the normal frame and coordinates, we have $L^i_{12} = 0, i = 1, 2$. The following expressions

$$\begin{aligned} L^1_{11} &= (\alpha + a)E, & L^2_{11} &= \beta E, \\ L^1_{12} &= 0, & L^2_{12} &= 0, \\ L^1_{22} &= (\alpha - a)G, & L^2_{22} &= \beta G \end{aligned}$$

hold. Let the Gauss curvature of F^2 be equal to -1. Then

$$\alpha^2 + \beta^2 - a^2 = -1.$$

Hence, $\alpha^2 - a^2 = -(1+\beta^2)$. We can write the expression for Gauss curvature K in terms of L^σ_{ij} as follows:

$$K = \frac{L^1_{11}L^1_{22} + L^2_{11}L^2_{22}}{EG} = \frac{L^1_{11}L^1_{22}}{EG} + \beta^2 = -1.$$

Therefore, we can write

$$\frac{L^1_{11}}{E\sqrt{1+\beta^2}} \frac{L^1_{22}}{G\sqrt{1+\beta^2}} = -1.$$

Denote $\frac{L^1_{11}}{E\sqrt{1+\beta^2}} = \operatorname{tg}\sigma$. Then $\frac{L^1_{22}}{G\sqrt{1+\beta^2}} = -\operatorname{ctg}\sigma$. Write the Codazzi equations in tensorial form as

$$L^\alpha_{ij,k} - L^\alpha_{ik,j} = \mu_{\sigma\alpha|k} L^\sigma_{ij} - \mu_{\sigma\alpha|j} L^\sigma_{ik},$$

where σ is the index of summation and $\mu_{\sigma\alpha|k}$ are the torsion coefficients. In developed form these equations take the forms

$$\frac{\partial L^\alpha_{ij}}{\partial u^k} - \frac{\partial L^\alpha_{ik}}{\partial u^j} - \Gamma^\beta_{ik} L^\alpha_{\beta j} + \Gamma^\beta_{ij} L^\alpha_{\beta k} = \mu_{\sigma\alpha|k} L^\sigma_{ij} - \mu_{\sigma\alpha|j} L^\sigma_{ik}.$$

Put $\alpha = 1$, $\sigma = 2$, $i = j = 1$, $k = 2$. Then the corresponding Codazzi equation is

$$\frac{\partial L^1_{11}}{\partial u^2} - \frac{\partial L^1_{12}}{\partial u^1} + \Gamma^2_{11} L^1_{22} - \Gamma^1_{12} L^1_{11} = \mu_{21|2} L^2_{11}.$$

As the coordinate system is orthogonal, the Christoffel symbols simplify to

$$\Gamma^2_{11} = -\frac{1}{2G}\frac{\partial E}{\partial v}, \quad \Gamma^1_{12} = \frac{1}{2E}\frac{\partial E}{\partial v}.$$

Recall that

$$L^1_{11} = \operatorname{tg}\sigma E\sqrt{1+\beta^2}, \quad L^1_{22} = -\operatorname{ctg}\sigma\sqrt{1+\beta^2}, \quad L^2_{11} = \beta E.$$

By substituting these expressions into the Codazzi equation, we get

$$\frac{\partial \operatorname{tg}\sigma E\sqrt{1+\beta^2}}{\partial u^2} + \frac{1}{2G}\frac{\partial E}{\partial u^2}\operatorname{ctg}\sigma G\sqrt{1+\beta^2} - \frac{1}{2E}\frac{\partial E}{\partial u^2}\operatorname{tg}\sigma E\sqrt{1+\beta^2} = \mu_{21|2}\beta.$$

The latter equation can be reduced to

$$\frac{\partial \operatorname{tg}\sigma\sqrt{1+\beta^2}}{\partial v} + (\operatorname{tg}\sigma + \operatorname{ctg}\sigma)\frac{E_v\sqrt{1+\beta^2}}{2E} = \mu_{21|2}\beta. \tag{2.1}$$

Put $\alpha = 2$, $\sigma = 1$, $i = l = 1$, $k = 2$. The corresponding Codazzi equation can be reduced to

$$\frac{\partial\beta}{\partial v} = \mu_{12|2}\operatorname{tg}\sigma\sqrt{1+\beta^2}. \tag{2.2}$$

Exclude $\mu_{12|2}$ from (2.2) and plug into (2.1). After some transformations, we get

$$\frac{\partial}{\partial v}\ln\left(\frac{\sqrt{E(1+\beta^2)}}{\cos\sigma}\right) = 0.$$

As a consequence, $\sqrt{E(1+\beta^2)} = C(u)\cos\sigma$. By changing the u-parameter, we can get $C(u) = 1$. Therefore, one can put $E(1+\beta^2) = \cos^2\sigma$. By using the other two Codazzi equations, we obtain $G(1+\beta^2) = \sin^2\sigma$.

Thus we can write the expressions for three fundamental quadratic forms:

$$ds^2 = \frac{\cos^2\sigma\, du^2 + \sin^2\sigma\, dv^2}{1+\beta^2},$$

$$II^1 = \frac{\sin\sigma\cos\sigma(du^2 - dv^2)}{\sqrt{1+\beta^2}}, \quad II^2 = \beta\, ds^2.$$

Let as pass to new new coordinates (p, q) by

$$u = p + q, \quad v = p - q.$$

Then ds^2 takes the conformal Chebyshev form and the coordinate lines $p = \text{const}$ and $q = \text{const}$ become asymptotic lines of the form II^1. Namely,

$$ds^2 = \frac{dp^2 + 2\cos\omega\, dp\, dq + dq^2}{1+\beta^2},$$

$$II^1 = \frac{2\sin\omega\, dp\, dq}{\sqrt{1+\beta^2}}, \qquad II^2 = \beta\, ds^2,$$

where $\omega = 2\sigma$.

Notice that the system of equations for isometric immersion of a 2-dimensional metric into 4-dimensional Euclidean space consists of one Gauss equation, four Codazzi equations and one Ricci equation (A.Sym and J.Cieslinski claimed that the latter equation was first derived by Kühne). In the case under consideration, we intend to show that the system can be reduced to two equations for two functions ω and β.

We begin with the Gauss equation. Introduce the metric

$$dl^2 = (1+\beta^2)\, ds^2.$$

Denote by K and K_l the Gauss curvatures of ds^2 and dl^2, respectively. Then

$$K_l = \frac{K - \nabla_2 \ln\sqrt{1+\beta^2}}{1+\beta^2},$$

where ∇_2 is the Laplace–Beltrami operator with respect to ds^2. In our case we can set $K = -1$. Denote by dS and dS_l the area elements for ds^2 and ds_l^2, respectively. Then

$$dS = \frac{\sin\omega}{1+\beta^2}\, dp\, dq, \quad dS_l = \sin\omega\, dp\, dq.$$

Over any domain $\Omega \subset F^2$ we have

$$\int_\Omega K_l\, dS_l = -\int_\Omega (1 + \nabla_2 \ln\sqrt{1+\beta^2})\, dS.$$

With respect to the (p,q)-coordinates, the curvature K_l can be easily calculated:

$$K_l = -\frac{\omega_{pq}}{\sin\omega}.$$

We have the equation

$$1 + \nabla_2 \ln\sqrt{1+\beta^2} = \frac{(1+\beta^2)\omega_{pq}}{\sin\omega}. \tag{2.3}$$

Hence,

$$\int_\Omega (1 + \nabla_2 \ln\sqrt{1+\beta^2})\, dS = \int_\Omega \omega_{pq}\, dp\, dq. \tag{2.4}$$

If Ω is the coordinate rectangle with vertices at P_i, then

$$\int_\omega \omega_{pq}\, dp\, dq = \sum_{i=1}^{4} \omega(P_i)(-1)^i. \tag{2.5}$$

Since F^2 is a regular surface, we have $0 < \omega(P_i) < \pi$. Therefore the module of the right-hand side of (2.5) is bounded from above by 2π.

On the Lobachevsky plane, consider the family of concentric disks C_r of radius r bounded by circles Γ_r. We have

$$\int_{C_r} \nabla_2 \ln\sqrt{1+\beta^2}\, dS = \int_{\Gamma_r} \frac{\partial}{\partial\nu}\left(\ln\sqrt{1+\beta^2}\right) ds,$$

where $\frac{\partial}{\partial\nu}$ is a derivative along the exterior normal to Γ_r and s is the arc length parameter of Γ_r.

Denote by D_r the image of the geodesic disc C_r in the (p,q)-plane endowed with the metric dl^2. Consider the integral

$$\int_{D_r} K_l\, dS_l = -\int_{D_r} \omega_{pq}\, dp\, dq.$$

Generally speaking, this integral is not bounded from above by a universal constant. However, for every bounded domain D_r there is some coordinate rectangle that covers D_r such that the integral of K_l over the rectangle is bounded from above by a universal constant.

In what follows, we will point out the conditions on dl^2 under which the integral of $-K_l$ over every bounded domain D is bounded from above by some universal constant M, i.e.,

$$-\int_D K_l\, dS_l < M = \text{const}.$$

Write the Lobachevsky metric with respect to the polar coordinates r, ϕ as

$$ds^2 = dr^2 + \operatorname{sh}^2 r\, d\phi^2.$$

The arc length element of Γ_r is $ds = \operatorname{sh} r\, d\phi$. Thus we have

$$\int_{\Gamma_r} \frac{\partial}{\partial r}\left(\ln\sqrt{1+\beta^2}\ \operatorname{sh} r\right) d\phi = \frac{d}{dr}\int_{\Gamma_r} \ln\sqrt{1+\beta^2}\, ds - \int_{\Gamma_r} \ln\sqrt{1+\beta^2}\ \operatorname{ch} r\, d\phi.$$

Denote by $S(r)$ the area of the geodesic disk C_r on the Lobachevsky plane. Then

$$S(r) + \frac{d}{dr}\int_{\Gamma_r} \ln\sqrt{1+\beta^2}\, ds - \int_{\Gamma_r} \ln\sqrt{1+\beta^2}\operatorname{ch} r\, d\phi = -\int_{D_r} K_l\, dS_l.$$

Denote $\theta = \ln\sqrt{1+\beta^2}$. Dividing both sides of the equation by $S(r)$, we get

$$1 + \frac{d}{dr}\left(\frac{1}{S}\int_{\Gamma_r}\theta\, ds\right) + \frac{S'}{S^2}\int_{\Gamma_r}\theta\, ds - \frac{\operatorname{ch} r}{S}\int_{\Gamma_r}\theta\, d\phi = -\frac{1}{S}\int_{D_r} K_l\, dS_l. \tag{2.6}$$

Notice that $S(r) = 2\pi(\operatorname{ch} r - 1)$, $S' = 2\pi \operatorname{sh} r$. Equation (2.6) takes the form

$$1 + \frac{d}{dr}\left(\frac{1}{S(r)}\int_{\Gamma_r}\theta\, ds\right) + \frac{\operatorname{sh}^2 r - \operatorname{ch} r(\operatorname{ch} r - 1)}{2\pi(\operatorname{ch} r - 1)^2}\int_{\Gamma_r}\theta\, d\phi = -\frac{1}{S(r)}\int_{D_r} K_l\, dS_l.$$

Suppose that the integral of $-K_l$ over each bounded domain is bounded from above by a constant M. Introduce the function

$$f(r) = \frac{1}{S(r)}\int_{\Gamma_r}\theta\, ds.$$

We get the inequality

$$f'(r) \le -1 - \frac{1}{2\pi(\operatorname{ch} r - 1)}\int_{\Gamma_r}\theta\, ds + \frac{M}{S(r)}.$$

The third term in the right-hand side of the inequality tends to zero when $r \to \infty$. Hence, the derivative of the function $f(r)$ becomes less than -1 for large enough r, and therefore the function $f(r)$ is negative for large enough r. But the function θ is always positive. We come to contradiction.

Consider now the conditions under which the absolute value of the integral of $-K_l$ is bounded. Note that dl^2 is a complete metric.

1) Let the curvature do not change the sign. For any geodesic disk C_r there exists a coordinate rectangle Ω that covers C_r. Then

$$\left|\int_{C_r} K_l\, dS_l\right| \le \left|\int_{\Omega} K_l\, dS_l\right| \le \left|\sum_{i=1}^{4}\omega(P_i)(-1)^i\right| < 2\pi.$$

2) Let the curvature K_l change the sign over a finite number of bounded domains. There exists a geodesic disk C_r that contains all these domains. Consider two cases:

a) The Gauss curvature $K_l \geq 0$ at infinity and over a finite number of bounded domains $K_l \leq 0$. Denote by Λ a union of all the domains with $K_l \leq 0$. We have

$$-\int_{C_r} K_l\, dS_l = -\int_{\Lambda} K_l\, dS_l - \int_{C_r-\Lambda} K_l\, dS_l. \tag{2.7}$$

The first term in the right-hand side of (2.7) is nonnegative but bounded from above by some number M since Λ consists of a finite number of domains. The second term is non-positive. Hence, for enough large r,

$$-\int_{C_r} K_l\, dS_l < M.$$

b) Suppose that $K_l \leq 0$ at infinity. Let the number of bounded domains with $K_l > 0$ be finite. Denote by Λ the union of all domains with $K_l > 0$. Let C_r be a geodesic disk which contains Λ. We can write again equation (2.7). Now the first term in the right-hand side of (2.7) is negative. Let Ω be the coordinate rectangle that contains C_r. We have the equation

$$-\int_{\Omega} K_l\, dS_l = -\int_{\Lambda} K_l\, dS_l - \int_{\Omega-\Lambda} K_l\, dS_l. \tag{2.8}$$

The left-hand side of (2.8) is bounded from above by 2π. The first term on the right-hand side is negative because $\Lambda \subset C_r \subset \Omega$ and is bounded in module by some number M. Therefore, the second term is also bounded from above by $M + 2\pi$, i.e.,

$$-\int_{\Omega-\Lambda} K_l\, dS_l \leq M + 2\pi.$$

But $C_r - \Lambda \subset \Omega - \Lambda$. Hence,

$$-\int_{C_r-\Lambda} K_l\, dS_l \leq -\int_{\Omega-\Lambda} K_l\, dS_l < M + 2\pi.$$

From (2.7) it follows that

$$-\int_{C_r} K_l < M_1 = \text{const}.$$

This inequality is valid for all large enough r. Therefore, in this case we also come to contradiction.

Theorem A is proved. □

The non-existence condition for isometric immersion of complete L^2 into E^4 can be formulated in terms of the function β. For example, if β satisfies

$$\nabla_2 \ln \sqrt{1+\beta^2} \geq (\epsilon - 1), \quad \epsilon > 0,$$

then the isometric immersion of complete L^2 into E^4 does not exist.

3. Fundamental system equations of isometric immersions of L^2 into E^4 with flat normal connection

We have already obtained the expression for the torsion coefficient

$$\mu_{12|2} = \frac{\beta_v \operatorname{ctg}\sigma}{\sqrt{1+\beta^2}}.$$

From one of the Codazzi equations we get

$$\mu_{12|1} = -\frac{\beta_u \operatorname{tg}\sigma}{\sqrt{1+\beta^2}}.$$

The Ricci (Kühne) equation has the form

$$\frac{\partial \mu_{12|1}}{\partial v} - \frac{\partial \mu_{12|2}}{\partial u} = 0.$$

Substitution of the torsion coefficient yields

$$\frac{\partial}{\partial v}\left(\frac{\beta_u \operatorname{tg}\sigma}{\sqrt{1+\beta^2}}\right) + \frac{\partial}{\partial u}\left(\frac{\beta_v \operatorname{ctg}\sigma}{\sqrt{1+\beta^2}}\right).$$

Denote $\rho = \ln(\beta + \sqrt{1+\beta^2})$. Then we come to the linear hyperbolic equation

$$\rho_{uv} + \rho_u \sigma_v \operatorname{tg}\sigma - \rho_v \sigma_u \operatorname{ctg}\sigma = 0$$

with respect to ρ. In terms of $\theta = \ln\sqrt{1+\beta^2}$ and $\gamma = \operatorname{arctg}\beta$ this equation can be written as

$$\left(\frac{\theta_p - \theta_q\cos\omega}{\sin\omega}\right)_p + \left(\frac{\theta_p\cos\omega - \theta_q}{\sin\omega}\right)_q = \frac{\gamma_p^2 - \gamma_q^2}{\sin\omega}.$$

The Gauss equation with respect to the metric of the Lobachevsky plane of curvature $K = -1$ takes the form

$$1 = \frac{1+\beta^2}{\sin\omega}\left\{\left(\frac{\theta_p - \theta_q\cos\omega}{\sin\omega}\right)_p + \left(\frac{\theta_q - \theta_p\cos\omega}{\sin\omega}\right)_q - \omega_{pq}\right\}.$$

Denote

$$\left(\frac{\theta_p - \theta_q\cos\omega}{\sin\omega}\right)_p = A, \qquad \left(\frac{\theta_p\cos\omega - \theta_q}{\sin\omega}\right)_q = B.$$

Then the system of equations for isomeric immersion of L^2 into E^4 with flat normal connection takes the form of two equations for two functions β and ω. Namely,

$$A + B = \frac{\gamma_p^2 - \gamma_q^2}{\sin\omega}, \qquad \gamma = \operatorname{arctg}\beta,$$

$$A - B = \omega_{pq} - \sin\omega e^{-2\theta}, \qquad \theta = \ln\sqrt{1+\beta^2}.$$

4. On local isometric immersions of L^2 into E^4 with flat normal connection and $\omega = \text{const}$

Theorem C. *There is no local isometric immersion of L^2 into E^4 with flat normal connection and $\omega = const$.*

Proof. We use now the equation for the function ρ in the (u, v)-coordinates. If $\omega = \text{const}$, then $\rho_{uv} = 0$, and hence $\rho = a(u) + b(v)$. Notice that $\beta = \operatorname{sh}\rho$. We have

$$ds^2 = \frac{1}{1+\beta^2}\left(\cos^2\sigma\,(du)^2 + \sin^2\sigma\,(dv)^2\right) = \frac{1}{1+\beta^2}\Big((d\cos\sigma u)^2 + (d\sin\sigma v)^2\Big).$$

Introduce new coordinates $x = u\cos\sigma$, $y = v\sin\sigma$. Then the coefficients of ds^2 take the form $E = G = \frac{1}{\operatorname{ch}^2\rho}$. The Gauss equation takes the form

$$K = \operatorname{sh}\rho\operatorname{ch}\rho(\rho_{xx} + \rho_{yy}) + \rho_x^2 + \rho_y^2.$$

Denote $\rho_x^2 = A(x)$, $\rho_{xx} = C(x)$, $\rho_y^2 = B(x)$, $\rho_{yy} = D(y)$. Then $A_x = 2\rho_x C$, $B_y = 2\rho_y D$. Suppose that $K = -1$. Write the Gauss equation as

$$\operatorname{sh} 2\rho = -2\frac{1+A+B}{C+D}.$$

The derivatives of both parts of this equation yield the equations

$$2\rho_x \operatorname{ch} 2\rho = -2\frac{A_x}{C+D} + 2\frac{(1+A+B)C_x}{(C+D)^2},$$
$$2\rho_y \operatorname{ch} 2\rho = -2\frac{B_y}{C+D} + 2\frac{(1+A+B)D_y}{(C+D)^2}.$$

Denote

$$\frac{C_x}{\rho_x} = L, \quad \frac{D_y}{\rho_y} = M.$$

We can write two expressions for $\operatorname{ch} 2\rho$:

$$\operatorname{ch} 2\rho = -\frac{2C}{C+D} + \frac{(1+A+B)L}{(C+D)^2},$$
$$\operatorname{ch} 2\rho = -\frac{2D}{C+D} + \frac{(1+A+b)M}{(C+D)^2}.$$

By using these equations, we get

$$-2\frac{C-D}{C+D} + \frac{(1+A+B)(L-M)}{(C+D)^2} = 0.$$

Suppose that $C + D \neq 0$. Then we have

$$2(C^2 - D^2) = (1+A+B)(L-M).$$

Differentiating first by x and then by y, we obtain the equation

$$\frac{A_x}{L_x} = \frac{B_y}{M_y} = k_0 = \mathrm{const}$$

with separable variables. Integrating, we get

$$A - k_0 L = k_1, \quad B - k_0 M = k_2,$$

where k_i are the constants of integration. Hence,

$$L = \frac{A - k_1}{k_0}, \quad M = \frac{B - k_0}{k_0}.$$

Thus we have come to the symmetric expression for $\operatorname{ch} 2\rho$:

$$\operatorname{ch} 2\rho = -1 + \frac{(1 + A + B)(A + B - k_1 - k_2)}{2k_0(C + D)^2}.$$

Besides,

$$2(C^2 - D^2) = (1 + A + B)\frac{(A - B + k_2 - k_1)}{k_0}.$$

Now we can separate the variables. We get one equation with the argument x,

$$2C^2 - \frac{A(1 + k_4)}{k_0} - \frac{A^2}{k_0} = k_5 = \mathrm{const}, \quad k_4 = k_2 - k_1,$$

and the other equation with the argument y. Take the derivative in x and use $C_x = \rho_x \frac{A-k_1}{k_0}$, $A_x = 2\rho_x C$. Then

$$4CC_x - A_x \frac{1 + k_4}{k_0} - 2AA_x \frac{1}{k_0} = 0.$$

In case of $C\rho_x \neq 0$, we get

$$4\frac{A - k_1}{k_0} - 2\frac{1 + k_4}{k_0} - 4\frac{A}{k_0} = 0.$$

It follows then that

$$k_1 + k_2 = -1.$$

The symmetric expression for $\operatorname{ch} 2\rho$ yields the equation

$$\operatorname{ch} 2\rho = -1 + \frac{1}{2k_0}\left(\frac{1 + A + B}{C + D}\right)^2 = -1 + \frac{\operatorname{sh}^2 2\rho}{8k_0}.$$

As a consequence, $\rho = const$, which contradicts to the Gauss equation. If $k_0 = 0$, then $A_x = B_y = 0$ and $\rho_x^2 + \rho_y^2 = -1$. In the case $C = 0$ or $D = 0$, we also come to contradiction. Theorem C is proved. $\square$

5. An example of Chebyshev metric with a sequence of bounded domains for which the integral curvature is unbounded from above

We intend to show that there is a metric

$$dl_2 = dp^2 + 2\cos\omega\, dp\, dq + dq^2$$

and a sequence Ω_n such that

$$\int_{\Omega_n} K_l dS_l \to \infty$$

when $n \to \infty$.

On the (p, q)-plane introduce the polar coordinates (r, ϕ). In the capacity of the domains Ω_n we take the concentric disks M_r of radius r bounded by the circles γ_r centered at the origin of coordinate system. We have

$$p = r\cos\phi, \qquad r = \sqrt{p^2 + q^2},$$
$$q = r\sin\phi, \qquad \phi = \operatorname{arctg}\frac{q}{p}.$$

Then we obtain

$$\frac{\partial r}{\partial p} = \cos\phi, \qquad \frac{\partial \phi}{\partial p} = -\frac{\sin\phi}{r},$$
$$\frac{\partial r}{\partial q} = \sin\phi, \qquad \frac{\partial \phi}{\partial q} = \frac{\cos\phi}{r}.$$

Rewrite the double integral over M_r in terms of the contour integral along γ_r,

$$J = \int_{M_r} \omega_{pq}\, dp\, dq = \frac{1}{2}\int_{\gamma_r} -\omega_p\, dp + \omega_q\, dq.$$

The derivatives of ω are of the form:

$$\frac{\partial \omega}{\partial p} = \omega_r \cos\phi - \omega_\phi \frac{\sin\phi}{r},$$
$$\frac{\partial \omega}{\partial q} = \omega_r \sin\phi + \omega_\phi \frac{\cos\phi}{r}.$$

We get

$$J = \frac{1}{2}\int_{\gamma_r} \Big(\omega_r\cos\phi - \omega_\phi\frac{\sin\phi}{r}\Big)\, d(r\cos\phi) + \Big(\omega_r\sin\phi + \omega_\phi\frac{\cos\phi}{r}\Big)\, d(r\sin\phi)$$
$$= \frac{r}{2}\int_{\gamma_r}\Big(\omega_r \sin 2\phi - \omega_\phi \frac{\cos 2\phi}{r}\Big)\, d\phi.$$

After transformations we obtain

$$J = \frac{r}{2}\frac{d}{dr}\int_{\gamma_r} \omega \sin 2\phi\, d\phi - \int_{\gamma_r} \omega\sin 2\phi\, d\phi.$$

Suppose

$$\omega(r,\phi) = \epsilon b(r)\sin 2\phi + \frac{\pi}{4}, \quad \epsilon > 0.$$

Choose the function $b(r)$ such that $|b(r)| < 1$ and three derivatives at the origin are equal to zero. Choose ϵ small enough to satisfy $0 < \omega < \pi$. Under these conditions the metric is regular. We have

$$J = \frac{1}{2}\epsilon r \frac{db(r)}{dr} \int_o^{2\pi} \sin^2 2\phi \, d\phi - \epsilon b(r) \int_{\gamma_r} \sin^2 2\phi \, d\phi.$$

Take the sequence

$$r_n = \sum_{k=1}^{n} \frac{1}{k}.$$

It is easy to construct a bounded regular function $b(r)$ satisfying

$$b(r_n) = 0 \quad \text{and} \quad b(r_n + \frac{1}{2(n+1)}) = \pm\frac{1}{2}.$$

Choose $+$ for odd n and $-$ for even ones. Since the distance between r_n and r_{n+1} tends to zero, $|b'| \to \infty$ for some sequence of points. Therefore, for some sequence of *disks* M_r the integral curvature is not bounded from above.

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References

[1] D.V. Bolotov, *On isometric immersions with flat normal connection of Lobachevsky space L^n into Euclidean space E^{n+m}*, Mat. Zametki **82** (2007), No. 1, 11–13 (Russian).

[2] A.A. Borisenko, *On the structure of multidimensional submanifolds with metric of revolution in Euclidean space,* Zh. Mat. Fiz. Anal. Geom. **15** (2019), No. 2, 192–202.

[3] E. Cartan, *Riemannian Geometry in an Orthogonal Frame,* Moscow State University, Moscow, 1960 (Russian); Engl. transl.: World Sci. Publ. Co. Pte. Ltd., Singapore, 2001.

[4] V.O. Gorkavyy and R. Posylaieva, *On the sharpness of one integral inequality for closed curves in R^4*, Zh. Mat. Fiz. Anal. Geom. **15** (2019), No 4, 502–509.

[5] E.R. Rozendorn, *Realization of the metric $ds^2 = du^2 + f(u)^2 dv^2$ into five-dimensional Euclidean space,* Dokl. Acad. Sci. Armenian SSR (1960), 197–198 (Russian).

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Про ізометричні занурення площини Лобачевського в чотиривимірний евклідів простір з плоскою нормальною зв'язністю

Yuriy Aminov

Згідно з теоремою Гільберта, площина Лобачевського L^2 не може бути ізометрично зануреною в E^3. Питання існування ізометричного занурення L^2 в E^4 залишається відкритим. Ми розглядаємо ізометричні занурення в E^4 з плоскою нормальною зв'язністю і знаходимо фундаментальну систему двох диференціальних рівнянь з частинними похідними другого порядку для двох функцій. Доведено теореми про неіснування ізометричних глобальних та локальних занурень за певних умов.

Ключові слова: ізометричне занурення, індикатриса, кривизна, асимптотична крива