Journal of Mathematical Physics, Analysis, Geometry
2021, Vol. 17, No. 2, pp. 163–174
doi: https://doi.org/10.15407/mag17.02.163

The Interaction of an Infinite Number of Eddy Flows for the Hard Spheres Model

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In the paper, the explicit approximate solutions of the Boltzmann equation for the hard spheres model are obtained. They have the form of function series of Maxwellians with coefficient functions of a spatial coordinate and time. Sufficient conditions for minimizing the uniform-integral error between the parts of the Boltzmann equation for the constructed distribution are obtained.

Key words: Boltzmann equation, hard spheres, eddy flows, infinite modal distribution

Mathematical Subject Classification 2010: 76P05, 45K05, 82C40, 35Q55

1. Statement of the problem

The Boltzmann kinetic equation plays an important role in the kinetic theory of gases. In this paper, we consider this equation for a model of hard spheres that describes particles of any gas which move translationally with a certain linear velocity, collide by the laws of classical mechanics and can not rotate. For this model, the equation has the form [1,3]

$$D(f) = Q(f, f), \tag{1.1}$$

where the left-hand side of the equation is the differential operator

$$D(f) \equiv \frac{\partial f}{\partial t} + \left(V, \frac{\partial f}{\partial x}\right), \tag{1.2}$$

(here (a, b) is the scalar product of the vectors a and b) and the right-hand side of (1.1) is the collision integral, which for the hard spheres model is as follows

$$\begin{aligned} Q(f, f) \equiv \frac{d^2}{2} \int_{R^3} dV_1 \int_{\Sigma} d\alpha |(V - V_1, \alpha)| \\ \times \Big[f(t, x, V_1')f(t, x, V') - f(t, x, V)f(t, x, V_1)\Big], \end{aligned} \tag{1.3}$$

where $f(t, x, V)$ is the distribution function of particles; the parameter $t \in \mathbb{R}$ is time; $V = (V^1, V^2, V^3)$ is a linear velocity; $\frac{\partial f}{\partial x}$ is the gradient of the function f

of the spatial coordinate $x \in \mathbb{R}^3$, which determines the location of the particle in space; $d > 0$ is the diameter of the molecule; α is the unit vector on the unit spheres Σ, and V, V_1, V', V_1' are the velocities of particles before and after collision, respectively, determined by the relations

$$V' = V - \alpha(V - V_1, \alpha),$$
$$V_1' = V_1 + \alpha(V - V_1, \alpha).$$

The only exact solution to equation (1.1), which is known explicitly up to now, is the Maxwellian, which makes both parts of the Boltzmann equation equal to zero. There are global Maxwellians, which, in the case of hard spheres, depend only on the velocity V, and there are local Maxwellians, which, unlike global ones, depend also on a spatial coordinate and time. The most general form of local Maxwellians for the hard spheres model was obtained in [1, 4, 7, 10].

In [9], a global Maxwellian was considered as well as a local Maxwellian, namely a screw, that depends on the spatial coordinate. We now consider Maxwellians, which also depend on time, i.e., non-stationary and inhomogeneous Maxwellians. From the physical point of view, they describe the motion of a gas rotating about a given axis and moving in the direction perpendicular to the axis. Analytically, these Maxwellians have the form [2, 4, 7, 10]

$$M_i(t, x, V) = \rho_i \left(\frac{\beta_i}{\pi}\right)^{3/2} e^{-\beta_i (V - \overline{V}_i)^2} \tag{1.4}$$

(here and in what follows, the index $i \in \mathbb{N}$) with the density

$$\rho_i = \rho_{0i} e^{\beta_i \omega_i^2 r_i^2}, \tag{1.5}$$

where ρ_{0i} is a nonnegative scalar constant, the parameter β_i is the quantity inverse to the absolute temperature

$$\beta_i = \frac{1}{2T_i}, \tag{1.6}$$

the vector ω_i is the angular velocity of the gas flow as a whole with which it rotates about some axis, and r_i^2 is the distance between the molecule and the axis of rotation x_{0i}

$$r_i^2 = \frac{1}{\omega_i^2}[\omega_i, x - x_{0i} - u_{0i}t]^2, \tag{1.7}$$

$$x_{0i} = \frac{1}{\omega_i^2}[\omega_i, \widehat{V}_i - u_{0i}], \tag{1.8}$$

(here $[a, b]$ is the vector product of the vectors a and b) the vector $u_{0i} \perp \omega_i$ and it is the linear velocity of the axis of the i−th rotating gas flow. By $\widehat{V}_i$, we denote the translational velocity of the flow included in the mass velocity

$$\overline{V}_i = \widehat{V}_i + [\omega_i, x - u_{0i}t]. \tag{1.9}$$

In this paper, we obtain the approximate solutions for the equation under study in the following form

$$f(t,x,V)=\sum_{i=1}^{\infty}\varphi_i(t,x)M_i(t,x,V). \tag{1.10}$$

The coefficient functions $\varphi_i(t,x)$ are nonnegative smooth functions on $\mathbb{R}^4$ and their norm

$$\|\varphi_i(t,x)\|=\sup_{(t,x)\in\mathbb{R}^4}\left(|\varphi_i(t,x)|+\left|\frac{\partial\varphi_i(t,x)}{\partial t}\right|+\left|\frac{\partial\varphi_i(t,x)}{\partial x}\right|\right) \tag{1.11}$$

is not equal to zero.

The aim of the work is to find the form of the coefficient functions $\varphi_i(t,x)$ and the conditions for the hydrodynamic parameters of Maxwellians for which the uniform-integral error [5]

$$\Delta=\Delta(\beta_i)=\sup_{(t,x)\in\mathbb{R}^4}\int_{\mathbb{R}^3}\big|D(f)-Q(f,f)\big|dV \tag{1.12}$$

can be arbitrarily small.

2. The main results

We now formulate and prove the next theorem. The parameter β_i is defined by the relation (1.6), and the function Δ has the form (1.12).

Theorem 2.1. *Let the coefficient functions have the form*

$$\varphi_i(t,x)=\psi_i(t,x)e^{-\beta_i\omega_i^2r_i^2}, \tag{2.1}$$

where $\psi_i(t,x)\geqslant 0$ are smooth nonnegative functions and their norm (1.11) is not equal to zero. Let all function series with one of the following common terms

$$\psi_i,\quad |x|\psi_i,\quad t\psi_i,\quad \left|\frac{\partial\psi_i}{\partial x}\right|,\quad \left|\frac{\partial\psi_i}{\partial t}\right|,\quad |x|\left|\frac{\partial\psi_i}{\partial x}\right|,\quad t\left|\frac{\partial\psi_i}{\partial t}\right| \tag{2.2}$$

converge uniformly on the $\mathbb{R}^4$ after multiplying by ρ_{0i}. Also let

$$\omega_i=\omega_{0i}\beta_i^{-m_i}, \tag{2.3}$$

where $m_i\geqslant\frac{1}{4}$.

Then there exists a function Δ' such that

$$\Delta\leqslant\Delta', \tag{2.4}$$

and

a) *if* $m_i > \frac{1}{2}$, *then*

$$\lim_{\beta_i \to +\infty} \Delta' = \sum_{i=1}^{\infty} \rho_{0i} \sup_{(t,x)\in\mathbb{R}^4} \left| \frac{\partial \psi_i}{\partial t} + \left(\widehat{V}_i, \frac{\partial \psi_i}{\partial x} \right) \right| + 2\pi d^2 \sum_{\substack{i,j=1\\ i\neq j}}^{\infty} \rho_{0i}\rho_{0j} \left| \widehat{V}_i - \widehat{V}_j \right| \sup_{(t,x)\in\mathbb{R}^4} (\psi_i \psi_j); \tag{2.5}$$

b) *if* $m_i = \frac{1}{2}$, *then in the right-hand side of* (2.5) *there is the additional term*

$$\frac{4}{\sqrt{\pi}} \sum_{i=1}^{\infty} \rho_{0i} \left| \left[\omega_{0i}, \widehat{V}_i - u_{0i} \right] \right| \sup_{(t,x)\in\mathbb{R}^4} \psi_i; \tag{2.6}$$

c) *if* $\frac{1}{4} < m_i < \frac{1}{2}$ *and vectors* $\omega_{0i}, \left(\widehat{V}_i - u_{0i} \right)$ *are parallel*

$$\omega_{0i} \parallel \left(\widehat{V}_i - u_{0i} \right), \tag{2.7}$$

then assertion (2.5) *is also true*;

d) *if* $m_i = \frac{1}{4}$ *and*

$$\omega_i = \omega_{0i} s_i \beta_i^{-\frac{1}{4}}, \tag{2.8}$$

where s_i *are positive constants and, in addition, require the validity of* (2.7), *then in the right-hand side of* (2.5) *there is the additional term*

$$\frac{4}{\sqrt{\pi}} \sum_{i=1}^{\infty} \rho_{0i} s_i^2 \omega_{0i}^2 \sup_{(t,x)\in\mathbb{R}^4} \left((|x| + |x - u_{0i}t|)\, \psi_i \right). \tag{2.9}$$

Remark 2.2. The notation $\beta_i \to +\infty$ means that for any number i there is the inequality $\beta_i > \overline{\beta}$, where $\overline{\beta} \to +\infty$.

Proof. The following inequality was obtained in details in [9]

$$\int_{\mathbb{R}^3} dV \left| D(f) - Q(f,f) \right| \leqslant \sum_{i=1}^{\infty} \int_{\mathbb{R}^3} dV M_i \left| \frac{\partial \varphi_i}{\partial t} + \left(V, \frac{\partial \varphi_i}{\partial x} \right) \right| + \widetilde{\mathbf{S}},$$

$$\widetilde{\mathbf{S}} = 2 \sum_{\substack{i,j=1\\ i\neq j}}^{\infty} \frac{d^2 \rho_i \rho_j}{\pi^2} \varphi_i \varphi_j \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dq_1 e^{-q^2 - q_1^2} \left| \frac{q}{\sqrt{\beta_i}} - \frac{q_1}{\sqrt{\beta_j}} + \overline{V}_i - \overline{V}_j \right|. \tag{2.10}$$

To the conditions imposed in [9], we add a condition of uniform convergence of all function series with a common term of one of the functions (2.2) additionally multiplied by a constant nonnegative factor ρ_{0i} that ensures the existence of (2.10). We use the derivatives of the coefficient functions obtained in [8]

$$\frac{\partial \varphi_i}{\partial t} = e^{-\beta_i \omega_i^2 r_i^2} \left(\frac{\partial \psi_i}{\partial t} + 2\beta_i \psi_i \left\{ \omega_i^2 (x, u_{0i}) - t\omega_i^2 u_{0i}^2 - \left([\omega_i, \widehat{V}_i], u_{0i} \right) \right\} \right), \tag{2.11}$$

$$\frac{\partial \varphi_i}{\partial x} = e^{-\beta_i \omega_i^2 r_i^2} \left(\frac{\partial \psi_i}{\partial x} + 2\beta_i \psi_i \left\{ \omega_i(\omega_i, x) - \omega_i^2 (x - x_{0i} - u_{0i} t) \right\} \right). \tag{2.12}$$

Let us substitute (2.11), (2.12) into inequality (2.10)

$$\begin{aligned} \int_{\mathbb{R}^3} dV \big| D(f) - Q(f,f) \big| \leqslant \sum_{i=1}^{\infty} \rho_i \left(\frac{\beta_i}{\pi} \right)^{3/2} \int_{\mathbb{R}^3} e^{-\beta_i (V - \overline{V})^2} \Bigg| \frac{\partial \psi_i}{\partial t} + \left(V, \frac{\partial \psi_i}{\partial x} \right) \\ + 2\beta_i \psi_i \left\{ \omega_i^2 (x, u_{0i}) - t \omega_i^2 u_{0i}^2 - \left([\omega_i, \widehat{V}_i], u_{0i} \right) \right\} \\ + \left(V, 2\beta_i \psi_i \left\{ \omega_i(\omega_i, x) - \omega_i^2 (x - x_{0i} - u_{0i} t) \right\} \right) \Bigg| e^{-\beta_i \omega_i^2 r_i^2} dV + \widetilde{\mathbf{S}}. \end{aligned}$$

Remembering the form (1.5) for the density of ρ_i, we can get the estimate

$$\begin{aligned} \int_{\mathbb{R}^3} dV \big| D(f) - Q(f,f) \big| \leqslant \sum_{i=1}^{\infty} \rho_{0i} \left(\frac{\beta_i}{\pi} \right)^{3/2} \int_{\mathbb{R}^3} dV e^{-\beta_i (V - \overline{V})^2} \Bigg| \frac{\partial \psi_i}{\partial t} + \left(V, \frac{\partial \psi_i}{\partial x} \right) \\ + 2\beta_i \psi_i \left\{ \omega_i^2 (x, u_{0i}) - t \omega_i^2 u_{0i}^2 - \left([\omega_i, \widehat{V}_i], u_{0i} \right) \right. \\ \left. + \ (V, \omega_i(\omega_i, x)) - \omega_i^2 (x - x_{0i} - u_{0i} t, V) \right\} \Bigg| + \widetilde{\mathbf{S}}. \end{aligned}$$

Now perform the change of variables

$$p = \sqrt{\beta_i} \left(V - \overline{V}_i \right) \Rightarrow V = \frac{p}{\sqrt{\beta_i}} + \overline{V}_i$$

with the Jacobian $\beta_i^{-3/2}$. Thus, we get

$$\begin{aligned} \int_{\mathbb{R}^3} dV \big| D(f) - Q(f,f) \big| \leqslant \pi^{-3/2} \sum_{i=1}^{\infty} \rho_{0i} \int_{\mathbb{R}^3} dp e^{-p^2} \Bigg| \frac{\partial \psi_i}{\partial t} + \left(\frac{p}{\sqrt{\beta_i}} + \overline{V}_i, \frac{\partial \psi_i}{\partial x} \right) \\ + 2\beta_i \psi_i \Bigg\{ \omega_i^2 (x, u_{0i}) - t \omega_i^2 u_{0i}^2 - \left([\omega_i, \widehat{V}_i], u_{0i} \right) \\ + \left(\frac{p}{\sqrt{\beta_i}} + \overline{V}_i, \omega_i(\omega_i, x) \right) - \omega_i^2 (x - x_{0i} - u_{0i} t, \frac{p}{\sqrt{\beta_i}} + \overline{V}_i) \Bigg\} \Bigg| + \widetilde{\mathbf{S}}, \end{aligned}$$

which can be written in the form

$$\begin{aligned} \int_{\mathbb{R}^3} dV \big| D(f) - Q(f,f) \big| \leqslant \pi^{-3/2} \sum_{i=1}^{\infty} \rho_{0i} \int_{\mathbb{R}^3} e^{-p^2} \Bigg| \frac{\partial \psi_i}{\partial t} + \left(\frac{p}{\sqrt{\beta_i}} + \overline{V}_i, \frac{\partial \psi_i}{\partial x} \right) \\ + 2\beta_i \psi_i \Bigg\{ \omega_i^2 (x, u_{0i}) - t \omega_i^2 u_{0i}^2 - \left([\omega_i, \widehat{V}_i], u_{0i} \right) \\ + \left(\widehat{V}_i, \omega_i \right) (\omega_i, x) - \left(\overline{V}_i, x - x_{0i} - u_{0i} t \right) \omega_i^2 \\ + \left(\frac{p}{\sqrt{\beta_i}}, \omega_i \right) (\omega_i, x) - \omega_i^2 \left(\frac{p}{\sqrt{\beta_i}}, x - x_{0i} - u_{0i} t \right) \Bigg\} \Bigg| dp + \widetilde{\mathbf{S}}. \end{aligned}$$

Let us show that under the sign of the integral, after the identical transformations, the addends with the first degree of the variable β_i vanish. Indeed,

$$\begin{aligned}
\omega_i^2(x,u_{0i}) - t\omega_i^2u_{0i}^2 - \left([\omega_i,\widehat{V}_i],u_{0i}\right) + \left(\widehat{V}_i,\omega_i\right)(\omega_i,x) \\
- \omega_i^2\left(\widehat{V}_i + [\omega_i, x-u_{0i}t], x - x_{0i} - u_{0i}t\right) \\
= \omega_i^2(x,u_{0i}) - t\omega_i^2u_{0i}^2 - \left([\omega_i,\widehat{V}_i],u_{0i}\right) + \left(\widehat{V}_i,\omega_i\right)(\omega_i,x) \\
- \omega_i^2\left(\widehat{V}_i, x - x_{0i} - u_{0i}t\right) + \left([\omega_i, x-u_{0i}t],[\omega_i,\widehat{V}_i - u_{0i}]\right) \\
= \omega_i^2(x,u_{0i}) - t\omega_i^2u_{0i}^2 - \left([\omega_i,\widehat{V}_i],u_{0i}\right) + \left(\widehat{V}_i,\omega_i\right)(\omega_i,x) \\
- \omega_i^2\left(\widehat{V}_i, x - u_{0i}t\right) + \left(\widehat{V}_i,[\omega_i,\widehat{V}_i - u_{0i}]\right) \\
+ \omega_i^2\left(x-u_{0i}t, \widehat{V}_i - u_{0i}\right) - \left(\omega_i,\widehat{V}_i\right)(x,\omega_i) \\
= \omega_i^2(x,u_{0i}) - t\omega_i^2u_{0i}^2 - \omega_i^2\left(\widehat{V}_i,x\right) + \omega_i^2\left(\widehat{V}_i,u_{0i}\right)t \\
+ \omega_i^2\left(x,\widehat{V}_i\right) - \omega_i^2(x,u_{0i}) - \omega_i^2\left(u_{0i},\widehat{V}_i\right)t + \omega_i^2u_{0i}^2t,
\end{aligned}$$

which is evidently equal to zero. Thus, we get

$$\begin{aligned}
&\int_{\mathbb{R}^3}\left|D(f)-Q(f,f)\right|dV \\
&\leqslant \pi^{-3/2}\sum_{i=1}^{\infty}\rho_{0i}\int_{\mathbb{R}^3}e^{-p^2}\left|\frac{\partial\psi_i}{\partial t} + \left(\frac{p}{\sqrt{\beta_i}} + \widehat{V}_i + [\omega_i,x-u_{0i}t], \frac{\partial\psi_i}{\partial x}\right)\right. \\
&\left. +2\sqrt{\beta_i}\psi_i\left((p,\omega_i)(\omega_i,x) - \omega_i^2(p,x-x_{0i}-u_{0i}t)\right)\right|dp + \widetilde{\mathbf{S}} \\
&\leqslant \pi^{-3/2}\sum_{i=1}^{\infty}\rho_{0i}\int_{\mathbb{R}^3}e^{-p^2}\left|\frac{\partial\psi_i}{\partial t} + \left(\frac{p}{\sqrt{\beta_i}} + \widehat{V}_i + [\omega_i,x-u_{0i}t], \frac{\partial\psi_i}{\partial x}\right)\right. \\
&\left. +2\sqrt{\beta_i}\psi_i\left((p,\omega_i)(\omega_i,x) - \omega_i^2(p,x-u_{0i}t)\right) + \left(p,\left[\omega_i,\widehat{V}_i - u_{0i}\right]\right)\right|dp + \widetilde{\mathbf{S}}.
\end{aligned}$$

The existence of a supremum with respect to a spatial coordinate and time is guaranteed by the assumption of the theorem that the series with common members of the form (2.2) converge uniformly over all admissible values. Then, passing to the supremum in the last inequality, we get

$$\begin{aligned}
\Delta \leqslant \Delta'\pi^{-3/2}\sum_{i=1}^{\infty}\rho_{0i}\int_{\mathbb{R}^3}e^{-p^2}\sup_{(t,x)\in\mathbb{R}^4}\left|\frac{\partial\psi_i}{\partial t} + \left(\frac{p}{\sqrt{\beta_i}} + \widehat{V}_i + [\omega_i,x-u_{0i}t], \frac{\partial\psi_i}{\partial x}\right)\right. \\
\left. + 2\sqrt{\beta_i}\psi_i\left((p,\omega_i)(\omega_i,x) - \omega_i^2(p,x-u_{0i}t)\right) + \left(p,\left[\omega_i,\widehat{V}_i - u_{0i}\right]\right)\right|dp \\
+ \frac{2d^2}{\pi^2}\sum_{\substack{i,j=1 \\ i\neq j}}^{\infty}\rho_{0i}\rho_{0j}
\end{aligned}$$

$$\times \sup_{(t,x)\in\mathbb{R}^4} \left(\psi_i \psi_j \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dq_1 e^{-q^2-q_1^2} \left| \frac{q}{\sqrt{\beta_i}} - \frac{q_1}{\sqrt{\beta_j}} + \overline{V}_i - \overline{V}_j \right| \right).$$

Applying condition (2.3) of the theorem, we obtain

$$\Delta \leqslant \Delta' = \pi^{-3/2} \sum_{i=1}^{\infty} \rho_{0i} \int_{\mathbb{R}^3} e^{-p^2} \sup_{(t,x)\in\mathbb{R}^4} \left| \frac{\partial \psi_i}{\partial t} \right.$$
$$+ \left(\frac{p}{\sqrt{\beta_i}} + \widehat{V}_i + \beta_i^{-m_i}[\omega_{0i}, x - u_{0i}t], \frac{\partial \psi_i}{\partial x} \right) + \beta_i^{-m_i} \left(p, \left[\omega_{0i}, \widehat{V}_i - u_{0i} \right] \right)$$
$$+ 2\sqrt{\beta_i}\psi_i \left(\beta_i^{-2m_i}(p,\omega_{0i})(\omega_{0i},x) - \omega_{0i}^2 \beta_i^{-2m_i}(p, x - u_{0i}t) \right) \Big| \, dp$$
$$+ \frac{2d^2}{\pi^2} \sum_{\substack{i,j=1\\ i\neq j}}^{\infty} \rho_{0i}\rho_{0j} \sup_{(t,x)\in\mathbb{R}^4} \left(\psi_i \psi_j \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dq_1 e^{-q^2-q_1^2} \left| \frac{q}{\sqrt{\beta_i}} - \frac{q_1}{\sqrt{\beta_j}} \right. \right.$$
$$\left. \left. + \widehat{V}_i - \widehat{V}_j + \beta_i^{-m_i}[\omega_{0i}, x - u_{0i}t] - \beta_j^{-m_j}[\omega_{0j}, x - u_{0j}t] \right| \right). \tag{2.13}$$

In order pass to the limit in the last equality, we first introduce the notation

$$\gamma_i = \frac{1}{\beta_i},$$

after which we have

$$\Delta \leqslant \Delta' = \pi^{-3/2} \sum_{i=1}^{\infty} \rho_{0i} \int_{\mathbb{R}^3} e^{-p^2} \sup_{(t,x)\in\mathbb{R}^4} \left(\left| \frac{\partial \psi_i}{\partial t} \right. \right.$$
$$+ \left(p\sqrt{\gamma_i} + \widehat{V}_i + \gamma_i^{m_i}[\omega_{0i}, x - u_{0i}t], \frac{\partial \psi_i}{\partial x} \right)$$
$$+ 2\psi_i \left(\gamma_i^{2m_i - \frac{1}{2}}(p,\omega_{0i})(\omega_{0i},x) - \omega_{0i}^2 \gamma_i^{2m_i-\frac{1}{2}}(p, x - u_{0i}t) \right)$$
$$\left. \left. + \gamma_i^{m_i - \frac{1}{2}} \left(p, \left[\omega_{0i}, \widehat{V}_i - u_{0i} \right] \right) \right| \, dp \right)$$
$$+ \frac{2d^2}{\pi^2} \sum_{\substack{i,j=1\\ i\neq j}}^{\infty} \rho_{0i}\rho_{0j} \sup_{(t,x)\in\mathbb{R}^4} \left(\psi_i \psi_j \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dq_1 e^{-q^2-q_1^2} \left| q\sqrt{\gamma_i} - q_1\sqrt{\gamma_j} \right. \right.$$
$$\left. \left. + \widehat{V}_i - \widehat{V}_j + \gamma_i^{m_i}[\omega_{0i}, x - u_{0i}t] - \gamma_j^{m_j}[\omega_{0j}, x - u_{0j}t] \right| \right). \tag{2.14}$$

The limiting passage in (2.13), as $\beta_i \to +\infty$, is equivalent to $\gamma_i \to +0$ in (2.14). It requires the continuity of expression (2.14) at zero provided by the condition of uniform convergence and an obvious estimate $|\gamma_i| \leqslant \frac{1}{\beta}$ (see Remark 2.2). Here we use the lemma from [6] about the continuity of the supremum with respect to the parameter and the theorems about the continuity of integral and function series with respect to the parameter. Performing the limit transition (2.13) for $m_i > \frac{1}{2}$, we obtain Proposition (2.5), and in case $m_i = \frac{1}{2}$, the term (2.6) appears. Considering the value m_i from the interval $\left(\frac{1}{4}, \frac{1}{2}\right)$ and basing on (2.14), we conclude that the low temperature limit Δ' exists only when (2.7) is applied. After minor transformations the low temperature limit Δ' coincides with the expression (2.5).

For the case $m_i = \frac{1}{4}$, let us change the condition (2.3) for (2.8). Then, substituting (2.8) into (2.14) and passing to the limit $\gamma_i \to +0$, which corresponds to $\beta_i \to +\infty$, we get the equality

$$\lim_{\beta_i \to +\infty} \Delta' = \pi^{-3/2} \sum_{i=1}^{\infty} \rho_{0i} \int_{\mathbb{R}^3} e^{-p^2} \sup_{(t,x)\in\mathbb{R}^4} \left(\left| \frac{\partial \psi_i}{\partial t} + \left(\widehat{V}_i, \frac{\partial \psi_i}{\partial x} \right) \right. \right. \\ \left. \left. + 2\psi_i \left(s_i^2(p, \omega_{0i})(\omega_{0i}, x) - \omega_{0i}^2 s_i^2(p, x - u_{0i}t) \right) \right| dp \\ + 2\pi d^2 \sum_{\substack{i,j=1 \\ i \neq j}}^{\infty} \rho_{0i}\rho_{0j} \left| \widehat{V}_i - \widehat{V}_j \right| \sup_{(t,x)\in\mathbb{R}^4} (\psi_i \psi_j)$$

that does not exceed

$$\sum_{i=1}^{\infty} \rho_{0i} \sup_{(t,x)\in\mathbb{R}^4} \left| \frac{\partial \psi_i}{\partial t} + \left(\widehat{V}_i, \frac{\partial \psi_i}{\partial x} \right) \right| + 2\pi d^2 \sum_{\substack{i,j=1 \\ i \neq j}}^{\infty} \rho_{0i}\rho_{0j} \left| \widehat{V}_i - \widehat{V}_j \right| \sup_{(t,x)\in\mathbb{R}^4} (\psi_i \psi_j) \\ + \frac{4}{\sqrt{\pi}} \sum_{i=1}^{\infty} \rho_{0i} s_i^2 \omega_{0i}^2 \sup_{(t,x)\in\mathbb{R}^4} \left((|x| + |x - u_{0i}t|) \psi_i \right).$$

Thus, all the statements of Theorem 2.1 are checked in details. □

We now give sufficient conditions for minimizing the deviation (1.12).

Corollary 2.3. *Let the functions $\psi_i(t, x)$ have the form*

$$\psi_i(t, x) = C_i \left(x - \widehat{V}_i t \right) \tag{2.15}$$

or

$$\psi_i(t, x) = E_i \left(\left[x, \widehat{V}_i \right] \right), \tag{2.16}$$

and let the functions C_i and E_i satisfy the conditions of Theorem 2.1. Also let one of the following conditions be true

$$\overline{V}_i = \overline{V}_j, \tag{2.17}$$

$$\operatorname{supp} \varphi_i \cap \operatorname{supp} \varphi_j = \emptyset \quad (i \neq j), \tag{2.18}$$

$$d \to 0. \tag{2.19}$$

Then the following assertions hold.

(i) *If $m_i > \frac{1}{2}$, then the error (1.12) can be made arbitrarily small.*

(ii) *If $m_i \in \left(\frac{1}{4}, \frac{1}{2}\right]$ and the condition (2.7) of Theorem 2.1 is fulfilled, then (1.12) is infinitesimally small.*

(iii) *If* $m_i = \frac{1}{4}$, *the condition* (2.7) *of Theorem* 2.1 *is fulfilled, and*

$$s_i \to +0, \tag{2.20}$$

then (1.12) *is infinitesimally small.*

Proof. Let us compute the derivatives of the functions $\psi_i(t,x)$ from (2.15)

$$\frac{\partial \psi_i}{\partial t} = -\left(\widehat{V}_i, C_i'\right), \quad \frac{\partial \psi_i}{\partial x} = C_i'. \tag{2.21}$$

If the functions $\psi_i(t,x)$ have the form (2.16), then

$$\frac{\partial \psi_i}{\partial t} = 0, \quad \frac{\partial \psi_i}{\partial x} = \left[E_i', \widehat{V}_i\right]. \tag{2.22}$$

As it is easy to see, for $m_i > \frac{1}{2}$, the derivatives (2.21) or (2.22) make equal to zero the first sum in (2.5), and under one of the conditions (2.17), (2.18) or (2.19), the second sum in the expression for the low-temperature limit Δ' is also equal to zero. For $m_i = \frac{1}{2}$, the condition of collinearity (2.7) vanishes the additional term (2.6). For $m_i = \frac{1}{4}$, (2.9) is equal to zero by the condition (2.20). □

Below is a theorem that contains another approach for obtaining coefficient functions in the distribution (1.10).

Theorem 2.4. *Let all function series with a common term of* (2.2) *after multiplying by a factor* $e^{\beta_i \omega_i^2 r_i^2}$ *retain the convergence uniformly on* $\mathbb{R}^4$. *Also let the condition* (2.3) *remain true for* $m_i \geqslant \frac{1}{2}$ *and* (2.7) *be valid.*

Then there exists a value Δ', *for which* (2.4) *is true, and*

$$\lim_{\beta_i \to +\infty} \Delta' = \sum_{i=1}^{\infty} \rho_{0i} \sup_{(t,x)\in\mathbb{R}^4} \left(\mu_i(t,x) \left| \frac{\partial \varphi_i}{\partial t} + \left(\widehat{V}_i, \frac{\partial \varphi_i}{\partial x}\right) \right| \right)$$
$$+ 2\pi d^2 \sum_{\substack{i,j=1 \\ i\neq j}}^{\infty} \rho_{0i}\rho_{0j} \left|\widehat{V}_i - \widehat{V}_j\right| \sup_{(t,x)\in\mathbb{R}^4} (\mu_i(t,x)\mu_j(t,x)\varphi_i\varphi_j), \tag{2.23}$$

where

$$\mu_i(t,x) = \begin{cases} e^{[\omega_{0i}, x - u_{0i}t]^2}, & m_i = \frac{1}{2} \\ 1, & m_i > \frac{1}{2} \end{cases}.$$

Proof. By substituting the Maxwellians M_i into the inequality (2.10), we have

$$\int_{\mathbb{R}^3} dV \left| D(f) - Q(f,f) \right|$$
$$\leqslant \sum_{i=1}^{\infty} \rho_{0i} \left(\frac{\beta_i}{\pi}\right)^{3/2} e^{\beta_i \omega_i^2 r_i^2} \int_{\mathbb{R}^3} dV e^{-\beta_i (V-\overline{V})^2} \left| \frac{\partial \varphi_i}{\partial t} + \left(V, \frac{\partial \varphi_i}{\partial x}\right) \right|$$
$$+ \frac{2d^2}{\pi^2} \sum_{\substack{i,j=1 \\ i\neq j}}^{\infty} \rho_{0i}\rho_{0j} e^{\beta_i \omega_i^2 r_i^2 + \beta_j \omega_j^2 r_j^2} \varphi_i \varphi_j$$

$$\times \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dq_1 e^{-q^2-q_1^2} \left| \frac{q}{\sqrt{\beta_i}} - \frac{q_1}{\sqrt{\beta_j}} + \overline{V}_i - \overline{V}_j \right|. \qquad (2.24)$$

The passage to the supremum with respect to the variables (t, x) is taken due to the condition of uniform convergence of the functional series. Then we use the inequality (2.4) to obtain

$$\Delta' = \sum_{i=1}^{\infty} \rho_{0i} \left(\frac{\beta_i}{\pi}\right)^{3/2} \sup_{(t,x)\in\mathbb{R}^4} \left(e^{\beta_i \omega_i^2 r_i^2} \int_{\mathbb{R}^3} dV e^{-\beta_i (V-\overline{V})^2} \left| \frac{\partial \varphi_i}{\partial t} + \left(V, \frac{\partial \varphi_i}{\partial x}\right) \right| \right)$$
$$+ \frac{2d^2}{\pi^2} \sum_{\substack{i,j=1 \\ i\neq j}}^{\infty} \rho_{0i}\rho_{0j} \sup_{(t,x)\in\mathbb{R}^4} \left(e^{\beta_i \omega_i^2 r_i^2 + \beta_j \omega_j^2 r_j^2} \varphi_i \varphi_j \right.$$
$$\left. \times \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dq_1 e^{-q^2-q_1^2} \left| \frac{q}{\sqrt{\beta_i}} - \frac{q_1}{\sqrt{\beta_j}} + \overline{V}_i - \overline{V}_j \right| \right). \qquad (2.25)$$

After performing the change of variables $p = \sqrt{\beta_i}\left(V - \overline{V}_i\right)$, we have

$$\Delta' = \sum_{i=1}^{\infty} \rho_{0i} \pi^{-3/2} \sup_{(t,x)\in\mathbb{R}^4} \left(e^{\beta_i \omega_i^2 r_i^2} \int_{\mathbb{R}^3} dp e^{-p^2} \left| \frac{\partial \varphi_i}{\partial t} + \left(\overline{V}_i + \frac{p}{\sqrt{\beta_i}}, \frac{\partial \varphi_i}{\partial x}\right) \right| \right)$$
$$+ \frac{2d^2}{\pi^2} \sum_{\substack{i,j=1 \\ i\neq j}}^{\infty} \rho_{0i}\rho_{0j} \sup_{(t,x)\in\mathbb{R}^4} \left(e^{\beta_i \omega_i^2 r_i^2 + \beta_j \omega_j^2 r_j^2} \varphi_i \varphi_j \right.$$
$$\left. \times \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dq_1 e^{-q^2-q_1^2} \left| \frac{q}{\sqrt{\beta_i}} - \frac{q_1}{\sqrt{\beta_j}} + \overline{V}_i - \overline{V}_j \right| \right). \qquad (2.26)$$

The passage to the low-temperature limit is taken in the same way as in the proof of Theorem 2.1 and we can calculate the limit of the new factor

$$\lim_{\beta_i\to+\infty} e^{\beta_i \omega_i^2 r_i^2} = \lim_{\beta_i\to+\infty} e^{\beta_i [\omega_i, x - x_{0i} - u_{0i} t]^2} = \lim_{\beta_i\to+\infty} e^{\beta_i [\omega_i, x - u_{0i} t]^2}$$
$$= \lim_{\beta_i\to+\infty} e^{\beta_i^{1-2m_i} [\omega_{0i}, x - u_{0i} t]^2} = \begin{cases} e^{[\omega_{0i}, x - u_{0i} t]^2} & m_i = \frac{1}{2} \\ 1, & m_i > \frac{1}{2} \end{cases}.$$

Further, computing the limit of (2.26) for $\beta_i \to +\infty$, we obtain the assertion of Theorem 2.4. □

3. Conclusions

Some approximate solutions of the Boltzmann equation for a model of hard spheres in the form of a function series with Maxwell modes that describe the eddy gas motion are constructed in the paper. The explicit form of the coefficient functions in the infinite-modal Maxwell distribution is obtained. For the constructed expressions, sufficient conditions for minimizing the uniform-integral error between the parts of the equation under consideration are found.

From a physical point of view, the constructed distribution describes the interaction of an unlimited number of eddy-like Maxwellian flows in a gas of hard spheres. These flows rotate around axes and move translationally. In this case, the rotation of all flows slows down simultaneously with the cooling of the gas. The solutions are approximate, but with an arbitrary degree of accuracy.

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Received February 14, 2020, revised September 1, 2020.

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Взаємодія нескінченного числа смерчоподібних течій для моделі твердих куль

O.O. Hukalov and V.D. Gordevskyy

У статті одержано наближені розв'язки рівняння Больцмана для моделі твердих куль у явному вигляді. Вони мають вигляд функціонального ряду максвелліанів з коефіцієнтними функціями просторової координати та часу. Одержано достатні умови мінімізації рівномірно-інтегрального відхилу між частинами рівняння Больцмана для побудованого розподілу.

Ключові слова: рівняння Больцмана, тверді кулі, смерчоподібні течії, нескінченно модальний розподіл