Journal of Mathematical Physics, Analysis, Geometry
2021, Vol. 17, No. 4, pp. 468–483
doi: https://doi.org/10.15407/mag17.04.468

On Landsberg Warped Product Metrics

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In this paper, we discuss a class of Finsler metrics which are called Finsler warped product metrics. These metrics were studied by B. Chen, Z. Shen and L. Zhao in 2018. Basically, we study the Berwald curvature of Finsler warped product metrics. Also, we characterize the Finsler warped product metrics of isotropic Berwald curvature, then we obtain that they are Randers metrics (Theorem 1.2). Moreover, we consider an important problem which is a unicorn problem in Finsler geometry for the class of Finsler metrics. In fact, we get the answer to the crucial question of this study whether such a Landsberg Finsler warped product metric is a Berwald metric or not (Theorem 1.3).

Key words: Finsler warped product metric, Landsberg metric, isotropic Berwald curvature

Mathematical Subject Classification 2010: 53B40, 53C60

1. Introduction

The warped product metrics form an important and rich class of metrics in Riemann–Finsler geometry and some interesting results were obtained in [2,8,9]. B. Chen, Z. Shen, and L. Zhao introduced a new class of Finsler metrics using the concept of the warped product structure on an n-dimensional manifold $M := I \times \breve{M}$, where I is an interval of $\mathbb{R}$ and $\breve{M}$ is an $(n-1)$-dimensional manifold equipped with a Riemannian metric [4]. In fact, it was considered in the following form:

$$F(u,v) = \breve{\alpha}(\breve{u},\breve{v})\phi\Big(u^1, \frac{v^1}{\breve{\alpha}(\breve{u},\breve{v})}\Big), \tag{1.1}$$

where $u = (u^1, \breve{u}), v = v^1 \frac{\partial}{\partial u^1} + \breve{v}$ and ϕ is a suitable function defined on a domain of $\mathbb{R}^2$. Throughout this paper, we always assume that the dimension of the product manifold $M := I \times \breve{M}$ is greater than two and our index conventions are as follows:

$$1 \le A \le B \le \cdots \le n, \quad 2 \le i \le j \le \cdots \le n.$$

The class of Finsler metrics can be concluded as the spherically symmetric Finsler metrics. It is necessary to mention that a Finsler metric F is said to be spherically symmetric if the orthogonal group $O(n)$ acts as isometries on F [12,15,18]. The formulas for the flag curvature and Ricci curvature of Finsler warped product metrics were obtained by B. Chen, Z. Shen, and L. Zhao in [4]. Also, these metrics were characterized as Einstein metrics [4]. H. Liu and X. Mo obtained the differential equation to characterize the metrics with vanishing Douglas curvature [10]. Moreover, H. Liu, X. Mo, and H. Zhang obtained equations that characterize metrics of constant flag curvature and then they constructed many new warped product Douglas metrics of constant Ricci [11].

The Cartan torsion C, the Berwald curvature B, the Landsberg curvature L, the S-curvature S, the χ-curvature χ, the H-curvature H, etc., are several important non-Riemannian quantities in Finsler geometry. Since they all vanish for Riemannian metrics, they are called non-Riemannian quantities (see [13, 17, 21]). F is called a Berwald metric if its Berwald curvature vanishes. A Finsler metric F on a manifold M is said to be of isotropic Berwald curvature if

$$B_{C\ DE}^{\ A} = \tau(u)(F_{v^C v^D}\delta^A_E + F_{v^C v^E}\delta^A_D + F_{v^D v^E}\delta^A_C + F_{v^C v^D v^E} v^A), \tag{1.2}$$

where $B_{C\ DE}^{\ A}$ are the coefficients of Berwald curvature and $\tau(u)$ is a scalar function on M.

X. Chen and Z. Shen proved that F is of isotropic Berwald curvature if and only if it is a Douglas metric with isotropic E-curvature, [5]. A. Tayebi and M. Rafie-Rad showed that if F is an isotropic Berwald Finsler metric, then F is of isotropic S-curvature [20]. A. Tayebi and B. Najafi proved that isotropic Berwald metrics of scalar flag curvature are of Randers type [19]. In [14], E. Peyghan and A. Tayebi proved that every generalized Berwald metric with non-zero scalar flag curvature or isotropic Berwald curvature is a Randers metric. In [6], G. Enli, H. Liu, and X. Mo showed that if F is a spherically symmetric Finsler metric of isotropic Berwald curvature, then F is a Randers metric. These studies motivated to consider the Berwald curvature and the isotropic Berwald curvature of Finsler warped product metrics. Moreover, H. Liu, and X. Mo characterized the Finsler warped product metrics to be Berwaldian. They obtained the following lemma:

Lemma 1.1 ([10]). *Let $F = \breve{\alpha}\phi(r,s)$ be a Finsler warped product metric, where $r = u^1$ and $s = \frac{v^1}{\breve{\alpha}}$. Then F is a Berwald metric if and only if*

$$\Phi = a(r)s^2 + b(r), \quad \Psi = c(r)s, \tag{1.3}$$

where $a = a(r)$, $b = b(r)$ and $c = c(r)$ are differentiable functions and Φ and Ψ are defined in (2.2).

In this paper, we prove the following important theorem:

Theorem 1.2. *Let $F = \breve{\alpha}\phi(r,s)$ be a Finsler warped product metric, where $r = u^1$ and $s = \frac{v^1}{\breve{\alpha}}$. Suppose that F is of isotropic Berwald curvature. Then F is a Randers metric.*

Recall that a Randers metric has to be of the form $F = \alpha + \beta$, which was studied by a physicist Randers in 1941 [16], where α is a Riemannian metric and β is a 1-form with $\|\beta\|_\alpha < 1$.

In local coordinates, the Landsberg curvature $L := L_{CDE}du^C \otimes du^D \otimes du^E$ is defined as $L_{CDE} := -\frac{1}{2}FF_{v^A}\frac{\partial^3 G^A}{\partial v^C \partial v^D \partial v^E}$. Note that every Berwald metric is a Landsberg metric. However, the converse is still an open problem in Finsler geometry called as "unicorn problem" [1]. Therefore, we also considered to study the unicorn problem for Finsler warped product metrics (see Section 6). In fact, we prove the following:

Theorem 1.3. *Let $F = \breve{\alpha}\phi(r,s)$ be a Finsler warped product metric, where $r = u^1$ and $s = \frac{v^1}{\breve{\alpha}}$. If F is a Landsberg metric, then either*

1. *F is Berwaldian or*

2. *there exist the smooth functions a_1, a_2, and a_4 of r such that*

$$F = \breve{\alpha}\left[\exp\left(\int_0^s \frac{ta_1(r) + 2a_2(r)}{t^2a_1(r) + 2ta_2(r) - 2a_4(r)}\,dt\right)\right]a_0(r), \tag{1.4}$$

where $a_0(r) = c\exp(\int a_1(r)dr)$ and c is a constant.

Notice that the metrics in (1.4) are singular. Actually, we prove that all regular Landsberg warped product metrics must be Berwaldian. Thus the unicorn problem cannot be successful in the class of Finsler metrics.

2. Preliminaries

Assume that F is a Finsler metric on an n-dimensional manifold M and in local coordinates $u^1, \ldots, u^n$ and $v = v^A\frac{\partial}{\partial v^A}$, $G = v^A\frac{\partial}{\partial u^A} - 2G^A\frac{\partial}{\partial v^A}$ is a spray induced by F. The spray coefficients G^A are defined by

$$G^A := \frac{1}{4}g^{AB}\{[F^2]_{u^C v^B}v^C - [F^2]_{u^B}\},$$

where $g_{AB}(u,v) = \left[\frac{1}{2}F^2\right]_{v^A v^B}$ and $(g^{AB}) = (g_{AB})^{-1}$. The spray coefficients of a Riemannian metric are determined by the Christoffel symbols as $\breve{G}^i(\breve{u},\breve{v}) = \frac{1}{2}\breve{\Gamma}^i_{jk}(\breve{u})\breve{v}^j\breve{v}^k$". The spray coefficients G^A of a Finsler warped product metric $F = \breve{\alpha}\phi(r,s)$ are given by [4],

$$G^1 = \Phi\breve{\alpha}^2, \quad G^i = \breve{G}^i + \Psi\breve{\alpha}^2\breve{l}^i, \tag{2.1}$$

where $\breve{l}^i = \frac{v^i}{\breve{\alpha}}$ and

$$\Phi = \frac{s^2(\omega_r\omega_{ss} - \omega_s\omega_{rs}) - 2\omega(\omega_r - s\omega_{rs})}{2(2\omega\omega_{ss} - \omega_s^2)}, \quad \Psi = \frac{s(\omega_r\omega_{ss} - \omega_s\omega_{rs}) + \omega_s\omega_r}{2(2\omega\omega_{ss} - \omega_s^2)}, \tag{2.2}$$

where $\omega = \phi^2$. Φ and Ψ can be simplified by Maple and given as follows:

$$\Phi = s\Psi + A, \tag{2.3}$$

$$\Psi = \frac{s\phi_r}{2\phi} - \frac{\phi_s}{\phi}A, \tag{2.4}$$

where

$$A := \frac{s\phi_{rs} - \phi_r}{2\,\phi_{ss}}. \tag{2.5}$$

The Berwald curvature $B = B_{C\ DE}^{\ A} du^C \otimes du^D \otimes du^E \otimes \frac{\partial}{\partial u^A}$ of a Finsler metric F is defined by

$$B_{C\ DE}^{\ A} := \frac{\partial^3 G^A}{\partial v^C \partial v^D \partial v^E}.$$

The Finsler metric F is called a Berwald metric if $B = 0$. Furthermore, F is said to be of isotropic Berwald curvature if its Berwald curvature $B_{C\ DE}^{\ A}$ satisfies

$$B_{C\ DE}^{\ A} = \tau(u)(F_{v^C v^D}\delta^A_E + F_{v^C v^E}\delta^A_D + F_{v^D v^E}\delta^A_C + F_{v^C v^D v^E} v^A), \tag{2.6}$$

where $\tau(u)$ is a scalar function. The E-curvature $E = E_{AB} du^A \otimes du^B$ of F is defined by

$$E_{AB} := \frac{1}{2}\frac{\partial^2}{\partial v^A \partial v^B}(\frac{\partial G^C}{\partial v^C}). \tag{2.7}$$

Moreover, F is said to have an isotropic E-curvature if there is a scalar function $\kappa = \kappa(u)$ on M such that

$$E = \frac{1}{2}(n+1)\kappa F^{-1} h, \tag{2.8}$$

where h is a family of bilinear forms $h_v = h_{AB} du^A \otimes du^B$ defined by $h_{AB} := FF_{v^A v^B}$.

By the definition, the Landsberg metrics are defined by

$$L_{CDE} := -\frac{1}{2} F F_{v^A} \frac{\partial^3 G^A}{\partial v^C \partial v^D \partial v^E} = 0. \tag{2.9}$$

Moreover,

$$D = D^A_{BCE} du^B \otimes du^C \otimes du^E$$

is a tensor on $TM \setminus \{0\}$ which is called the Douglas tensor, where

$$D^A_{BCE} := \frac{\partial^3}{\partial v^B \partial v^C \partial v^E}\Big(G^A - \frac{1}{n+1}\frac{\partial G^M}{\partial v^M} v^A\Big). \tag{2.10}$$

The Finsler metric F is called a Douglas metric if $D = 0$, equivalently, a Finsler metric is a Douglas metric if and only if $G^A v^B - G^B v^A$ are homogeneous polynomials in y of degree 3 (see [3]). Clearly, for a Berwald metrics, the spray coefficients G^i are quadratic in y. It follows that $D = 0$, (2.10). The Berwald metrics are Douglas metric. H. Liu and X. Mo proved in [10] that a warped product Finsler metric $F = \breve{\alpha}\phi(r,s)$ is of Douglas type if and only if

$$\Phi - s\Psi = \xi(r)s^2 + \eta(r), \tag{2.11}$$

where $\xi = \xi(r)$ and $\eta = \eta(r)$ are two differential functions.

In Section 5, we need the following lemma:

Lemma 2.1 ([5]). *For a Finsler metric F on an n-dimensional manifold M, the followings are equivalent:*

(a) *F is of isotropic Berwald curvature,*

(b) *$D_B{}^A{}_{CD} = 0$ and $E_{AB} = \frac{n+1}{2}\kappa F_{v^A v^B}$ for a scalar function $\kappa = \kappa(u)$ on M.*

3. E-curvature of Finsler warped product metrics

In this section, first we compute the E-curvature of a Finsler warped product metric $F = \breve{\alpha}\phi(r,s)$. Then we classify the metrics with isotropic E-curvature.

The following identities are obvious for a Finsler warped product metric $F = \breve{\alpha}\phi(r,s)$:

$$\breve{\alpha}_{v^1} = 0, \quad s_{v^1} = \frac{1}{\breve{\alpha}}, \quad s_{v^j} = -\frac{s\breve{l}_j}{\breve{\alpha}}, \quad \breve{\alpha}^2_{v^j} = 2\breve{\alpha}\breve{l}_j, \quad \breve{l}_m\breve{l}^m = 1, \tag{3.1}$$

where $\breve{l}_j := \breve{\alpha}_{v^j}$.

By (2.1), (2.7) and (3.1), we get

$$\begin{aligned} E &= E_{AB}du^A \otimes du^B \\ &= E_{11}du^1 \otimes du^1 + E_{1j}du^1 \otimes du^j + E_{i1}du^i \otimes du^1 + E_{ij}du^i \otimes du^j, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} E_{11} &= \frac{1}{2}\Big[\frac{\partial}{\partial v^1 \partial v^1}\Big(\frac{\partial G^1}{\partial v^1}\Big) + \frac{\partial}{\partial v^1 \partial v^1}\Big(\frac{\partial G^m}{\partial v^m}\Big)\Big] \\ &= \frac{1}{2\breve{\alpha}}\big[(n-2)\Psi_{ss} - s\Psi_{sss} + \Phi_{sss}\big], \end{aligned} \tag{3.3}$$

$$\begin{aligned} E_{1j} &= \frac{1}{2}\Big[\frac{\partial}{\partial v^1 \partial v^j}\Big(\frac{\partial G^1}{\partial v^1}\Big) + \frac{\partial}{\partial v^1 \partial v^j}\Big(\frac{\partial G^m}{\partial v^m}\Big)\Big] \\ &= \frac{-s}{2\breve{\alpha}}\big[(n-2)\Psi_{ss} - s\Psi_{sss} + \Phi_{sss}\big]\breve{l}_j, \end{aligned} \tag{3.4}$$

$$\begin{aligned} E_{i1} &= \frac{1}{2}\Big[\frac{\partial}{\partial v^i \partial v^1}\Big(\frac{\partial G^1}{\partial v^1}\Big) + \frac{\partial}{\partial v^i \partial v^1}\Big(\frac{\partial G^m}{\partial v^m}\Big)\Big] \\ &= \frac{-s}{2\breve{\alpha}}\big[(n-2)\Psi_{ss} - s\Psi_{sss} + \Phi_{sss}\big]\breve{l}_i, \end{aligned} \tag{3.5}$$

$$\begin{aligned} E_{ij} &= \frac{1}{2}\Big[\frac{\partial}{\partial v^i \partial v^j}\Big(\frac{\partial G^1}{\partial v^1}\Big) + \frac{\partial}{\partial v^i \partial v^j}\Big(\frac{\partial G^m}{\partial v^m}\Big)\Big] \\ &= \frac{1}{2\breve{\alpha}}\Big\{s^2\big[(n-2)\Psi_{ss} - s\Psi_{sss} + \Phi_{sss}\big]\breve{l}_i\breve{l}_j \\ &\quad + \big[n(\Psi - s\Psi_s) + s^2\Psi_{ss} + \Phi_s - s\Phi_{ss}\big]\breve{h}_{ij}\Big\}, \end{aligned} \tag{3.6}$$

where $\breve{h}_{ij} := \breve{\alpha}(\breve{l}_i)_{v^j}$. Now we prove the following proposition:

Proposition 3.1 ([7]). *The warped product metric $F = \breve{\alpha}\phi(r,s)$ is of isotropic E-curvature if and only if*

$$n(\Psi - s\Psi_s) + s^2\Psi_{ss} + \Phi_s - s\Phi_{ss} = (n+1)k(\phi - s\phi_s), \tag{3.7}$$

where $k = k(u)$ is a scalar function on M.

Proof. A direct computation yields

$$F_{v^1} = \phi_s, \quad F_{v^i} = (\phi - s\phi_s)\breve{l}_i.$$

By the above equations, we obtain

$$F_{v^1v^1} = \frac{1}{\breve{\alpha}}\phi_{ss}, \tag{3.8}$$

$$F_{v^1v^j} = -\frac{s\phi_{ss}}{\breve{\alpha}}\breve{l}_j, \tag{3.9}$$

$$F_{v^iv^1} = -\frac{s\phi_{ss}}{\breve{\alpha}}\breve{l}_i, \tag{3.10}$$

$$F_{v^iv^j} = \frac{1}{\breve{\alpha}}\big[s^2\phi_{ss}\breve{l}_i\breve{l}_j + (\phi - s\phi_s)\breve{h}_{ij}\big]. \tag{3.11}$$

By (2.8), we have

$$\frac{\partial}{\partial v^B}\frac{\partial}{\partial v^A}\left(\frac{\partial G^C}{\partial v^C}\right) = (n+1)kF_{v^Av^B}. \tag{3.12}$$

Suppose that F is of isotropic E-curvature. By (3.3)–(3.6), (3.8)–(3.11), and (3.12), we obtain

$$(n-2)\Psi_{ss} - s\Psi_{sss} + \Phi_{sss} = (n+1)k\phi_{ss}, \tag{3.13}$$

$$s^2\big[(n-2)\Psi_{ss} - s\Psi_{sss} + \Phi_{sss}\big]\breve{l}_i\breve{l}_j + \big[n(\Psi - s\Psi_s) + s^2\Psi_{ss} + \Phi_s - s\Phi_{ss}\big]\breve{h}_{ij} = (n+1)k\big[s^2\phi_{ss}\breve{l}_i\breve{l}_j + (\phi - s\phi_s)\breve{h}_{ij}\big]. \tag{3.14}$$

Substituting (3.13) into (3.14), we get

$$n(\Psi - s\Psi_s) + s^2\Psi_{ss} + \Phi_s - s\Phi_{ss} = (n+1)k(\phi - s\phi_s). \tag{3.15}$$

Conversely, assume that (3.15) holds. Differentiating (3.15) with respect to s, we obtain (3.13). By (3.13) and (3.15), (3.14) holds. Hence, we conclude that $F = \breve{\alpha}\phi(r, s)$ has an isotropic E-curvature if and only if (3.15) holds. This completes the proof of Proposition 3.1. □

4. Berwald curvature

In this section, we prove the following theorem:

Theorem 4.1. *Let $F = \breve{\alpha}\phi(r, s)$ be a Finsler warped product metric, where $r = u^1$ and $s = \frac{v^1}{\breve{\alpha}}$. If F is a Berwald metric, then one of the following holds:*

1. *F is a Riemannian metric or*
2. *F has the form*

$$F = \breve{\alpha}\Upsilon\big[s^2 e^{4(\int m(r)dr)}\big], \tag{4.1}$$

where Υ is any differentiable function.

Proof. Suppose that F is a Berwald metric. Substituting (2.3) into (1.3), we obtain

$$A = m(r)s^2 + b(r), \tag{4.2}$$
$$\Psi = c(r)s, \tag{4.3}$$

where $m(r) = a(r) - c(r)$. Plugging (4.3) into (2.4), we have

$$\frac{s\phi_r}{2\phi} - \frac{\phi_s}{\phi}A = c(r)s.$$

Then plugging (4.2) into the above equation yields

$$\frac{s\phi_r}{2\phi} - \frac{\phi_s}{\phi}[m(r)s^2 + b(r)] = c(r)s. \tag{4.4}$$

By (4.4), we have

$$2[m(r)s^2 + b(r)]\phi_s - s\phi_r + 2sc(r)\phi = 0. \tag{4.5}$$

By differentiating (4.5) with respect to the variable s, we have

$$2[m(r)s^2 + b(r)]\phi_{ss} - s\phi_{rs} - \phi_r + 2c(r)\phi + 2s[2m(r) + c(r)]\phi_s = 0. \tag{4.6}$$

Using (2.5) and (4.2), we obtain

$$m(r)s^2 + b(r) = \frac{s\phi_{rs} - \phi_r}{2\,\phi_{ss}}. \tag{4.7}$$

By (4.7), we have

$$2[m(r)s^2 + b(r)]\phi_{ss} - s\phi_{rs} + \phi_r = 0. \tag{4.8}$$

By (4.6)–(4.8), we have

$$s[2m(r) + c(r)]\phi_s - \phi_r + c(r)\phi = 0. \tag{4.9}$$

Multiplying (4.9) by s and then subtracting (4.5), we get

$$[c(r)s^2 - 2b(r)]\phi_s = c(r)s\phi. \tag{4.10}$$

Case I: $c(r)s^2 - 2b(r) \neq 0$. Then, by (4.10), it follows that

$$d(\ln\phi)_s = \frac{1}{2}d[\ln(c(r)s^2 - 2b(r))]_s.$$

Thus,

$$\phi = \gamma(r)\sqrt{c(r)s^2 - 2b(r)},$$

where $\gamma(r)$ is any positive smooth function. In this case, the corresponding warped product Finsler metric is a Riemannian metric.

Case II: $c(r)s^2 - 2b(r) = 0$. Note that $\phi > 0$ and $s \neq 0$. In this case, (4.10) is equivalent to

$$c(r) = 0, \quad c(r)s^2 - 2b(r) = 0. \tag{4.11}$$

By (4.11), it follows that

$$c(r) = 0, \quad b(r) = 0. \tag{4.12}$$

Plugging (4.12) into (4.9) yields

$$2sm(r)\phi_s - \phi_r = 0. \tag{4.13}$$

In this case, we solve only (4.13). The characteristic equation of (4.13) is

$$\frac{dr}{-1} = \frac{ds}{2sm(r)},$$

which is equivalent to

$$\frac{ds}{dr} = -2sm(r). \tag{4.14}$$

Hence, the solution of (4.13) is

$$\phi = \Upsilon[s^2 e^{4(\int m(r)dr)}],$$

where $\Upsilon(\cdot)$ is any differentiable function [22, Lemma 4.1]. □

5. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. Firstly, we prove the following proposition:

Proposition 5.1. *Let $F = \breve{\alpha}\phi(r, s)$ be a Finsler warped product metric, where $r = u^1$ and $s = \frac{v^1}{\breve{\alpha}}$. Then F is a Douglas metric with isotropic E-curvature if and only if*

$$\Psi = k(u)\phi + sd(r), \quad A = \xi(r)s^2 + \eta(r), \tag{5.1}$$

where Ψ and A are defined by (2.4) and (2.5), respectively.

Proof. Let $F = \breve{\alpha}\phi(r, s)$ be a Finsler warped product metric. Suppose that F is a Douglas metric with isotropic E-curvature. By [10, Lemma 3.3], F has vanishing Douglas curvature if and only if

$$\Phi - s\Psi = \xi(r)s^2 + \eta(r). \tag{5.2}$$

By Proposition 3.1, F is of isotropic E-curvature if and only if

$$n(\Psi - s\Psi_s) + s^2\Psi_{ss} + \Phi_s - s\Phi_{ss} = (n + 1)k(\phi - s\phi_s). \tag{5.3}$$

By (2.3), it is easy to see that (5.2) and (5.3) are equivalent to

$$A = \xi(r)s^2 + \eta(r), \tag{5.4}$$

$$(n+1)(\Psi - s\Psi_s) + A_s - sA_{ss} = (n+1)k(\phi - s\phi_s). \tag{5.5}$$

Plugging (5.4) into (5.5), we get

$$\Psi - s\Psi_s = k(\phi - s\phi_s). \tag{5.6}$$

Let $\Psi = s\bar{\Psi}$ and $\phi = s\bar{\phi}$. Then $\Psi - s\Psi_s = -s^2\bar{\Psi}_s$ and $\phi - s\phi_s = -s^2\bar{\phi}_s$. Plugging these two equations into (5.6), we obtain

$$\bar{\Psi}_s - k\bar{\phi}_s = 0. \tag{5.7}$$

Therefore,

$$\bar{\Psi} - k\bar{\phi} = d(r). \tag{5.8}$$

Thus,

$$\Psi = k\phi + sd(r). \tag{5.9}$$

Conversely, suppose that (5.4) and (5.9) hold. Note that the equation given by (5.4) is equivalent to (5.2). By (5.4) and (5.9), (5.5) holds. Hence, we obtain that F is a Douglas metric with isotropic E-curvature. □

Hence, we have the following theorem:

Theorem 5.2. *Let $F = \breve{\alpha}\phi(r,s)$ be a Finsler warped product metric, where $r = u^1$ and $s = \frac{v^1}{\breve{\alpha}}$. Then F is of isotropic Berwald curvature if and only if*

$$\Psi = k(u)\phi + sd(r), \tag{5.10}$$

$$A = \xi(r)s^2 + \eta(r), \tag{5.11}$$

where Ψ and A are defined by (2.4) and (2.5), respectively.

Proof. By Lemma 2.1 and Proposition 5.1, we get the proof of Theorem 5.2. □

Now we prove Theorem 1.2:

Proof of Theorem 1.2. Suppose that F is of isotropic Berwald curvature, that is, (5.10) and (5.11) hold. The using of (2.4) and (5.10) yields

$$\frac{s\phi_r}{2\phi} - \frac{\phi_s}{\phi}A = k\phi + sd(r).$$

Plug (5.11) into the above equation to obtain

$$\frac{s\phi_r}{2\phi} - \frac{\phi_s}{\phi}[\xi(r)s^2 + \eta(r)] = k\phi + sd(r). \tag{5.12}$$

By (5.12), it follows that

$$2[\xi(r)s^2+\eta(r)]\phi_s - s\phi_r + 2sd(r)\phi + 2k\phi^2 = 0. \tag{5.13}$$

Taking the derivative with respect to the variable s, we get

$$\begin{aligned} 2[\xi(r)s^2+\eta(r)]\phi_{ss} - s\phi_{rs} - \phi_r + 2d(r)\phi \\ + 2s[2\xi(r)+d(r)]\phi_s + 4k\phi\phi_s = 0. \end{aligned} \tag{5.14}$$

On the other hand, by (2.5) and (5.11), we have

$$\xi(r)s^2+\eta(r) = \frac{s\phi_{rs}-\phi_r}{2\,\phi_{ss}}. \tag{5.15}$$

By (5.15), it follows that

$$2[\xi(r)s^2+\eta(r)]\phi_{ss} - s\phi_{rs} + \phi_r = 0. \tag{5.16}$$

By (5.14)–(5.16), we have

$$s[2\xi(r)+d(r)]\phi_s - \phi_r + d(r)\phi + 2k\phi\phi_s = 0. \tag{5.17}$$

Multiplying (5.17) by s and then subtracting (5.13), we have

$$[d(r)s^2 + 2ks\phi - 2\eta(r)]\phi_s = d(r)s\phi + 2k\phi^2. \tag{5.18}$$

If $ds^2 - 2\eta \neq 0$, then the solution of (5.18) is given by [6, Theorem 4.2],

$$\phi = \frac{2ks + \sqrt{(4k^2+\sigma d(r))s^2 - \sigma\eta(r)}}{\sigma}.$$

Note that $F = \breve{\alpha}\phi(r,s), r = u^1, s = \frac{v^1}{\breve{\alpha}}$. It follows that

$$F = \frac{2kv^1 + \sqrt{(4k^2+\sigma d(r))(v^1)^2 - \sigma\eta(r)\breve{\alpha}^2}}{\sigma}. \tag{5.19}$$

Therefore F is a Randers metric. □

In [4, Lemma 3.1], B.Chen, Z.Shen and L.Zhao proved that a spherically symmetric metric is a Finsler warped product metric. Hence, the following corollary of Theorem 1.2 is obvious.

Corollary 5.3. *[6] Let $(\mathbb{B}^n(\upsilon), F)$ be a spherically symmetric Finsler manifold. Suppose that F is of isotropic Berwald curvature. Then F is a Randers metric.*

X.Cheng and Z.Shen proved in [5] that if F is a Finsler metric on a manifold M of dimension $n \geq 3$ with the conditions

$$D_C{}^A{}_{DE} = 0 \quad \textit{and} \quad L_{ACD} + cFC_{ACD} = 0,$$

for a scalar function $c = c(u)$ on M, then F is of isotropic Berwald curvature satisfying (2.6) for a scalar function $\tau = \tau(u)$ on M. In this case, F_u is not Euclidean where $\tau(u) = c(u)$ at a point u. Therefore, the following corollary is obviously given as a result of the Theorem 1.2:

Corollary 5.4. *Let $F = \breve{\alpha}\phi(r,s), r = u^1, s = \frac{v^1}{\breve{\alpha}}$ be a Douglas warped product metric on a manifold M of dimension $n \geq 3$. If $L_{ACD} + cFC_{ACD} = 0$, then F is a Randers metric.*

6. Proof of Theorem 1.3

A direct computation yields

$$F_{v^1} = \phi_s, \quad F_{v^i} = (\phi - s\phi_s)\breve{l}_i.$$

By (2.1) and the Landsberg curvature formula $L_{CDE} = -\frac{1}{2} F F_{v^A} \frac{\partial^3 G^A}{\partial v^C \partial v^D \partial v^E}$, we get

$$L_{111} = -\frac{\phi}{2}\Big[\phi_s \Phi_{sss} + (\phi - s\phi_s)\Psi_{sss}\Big], \tag{6.1}$$

$$L_{11i} = \frac{s\phi}{2}\Big[\phi_s \Phi_{sss} + (\phi - s\phi_s)\Psi_{sss}\Big]\breve{l}_i, \tag{6.2}$$

$$\begin{aligned} L_{1ij} = \frac{\phi}{2}\Big\{&\Big[\phi_s(\Phi_s - s\Phi_{ss} - s^2\Phi_{sss}) - s(\phi - s\phi_s)(\Psi_{ss} + s\Psi_{sss})\Big]\breve{l}_i\breve{l}_j \\ &- \Big[\phi_s(\Phi_s - s\Phi_{ss}) - s(\phi - s\phi_s)\Psi_{ss}\Big]\breve{a}_{ij}\Big\}, \end{aligned} \tag{6.3}$$

$$\begin{aligned} L_{ijk} = \frac{\phi}{2}\Big\{&\Big[s^3\phi_s\Phi_{sss} - 3s\phi_s(\Phi_s - s\Phi_{ss}) - (\phi - s\phi_s)[-3s^2\Psi_{ss} \\ &- s^3\Psi_{sss}]\Big]\breve{l}_i\breve{l}_j\breve{l}_k + [s\phi_s(\Phi_s - s\Phi_{ss}) \\ &- (\phi - s\phi_s)s^2\Psi_{ss}]\breve{l}_k\breve{a}_{ij}(i \to j \to k \to i)\Big\}, \end{aligned} \tag{6.4}$$

where we use (3.1).

Note that a Finsler metric F is called Landsberg metric if the Landsberg curvature is zero:

$$L_{CDE} = 0.$$

So, a Finsler warped product metric $F = \breve{\alpha}\phi(r, s)$ is a Landsberg metric if and only if Φ and Ψ satisfy

$$-\frac{\phi}{2}\Big[\phi_s \Phi_{sss} + (\phi - s\phi_s)\Psi_{sss}\Big] = 0, \tag{6.5}$$

$$\frac{s\phi}{2}\Big[\phi_s \Phi_{sss} + (\phi - s\phi_s)\Psi_{sss}\Big]\breve{l}_i = 0, \tag{6.6}$$

$$\begin{aligned} -\frac{\phi}{2}\Big\{&-\Big[\phi_s(\Phi_s - s\Phi_{ss} - s^2\Phi_{sss}) - s(\phi - s\phi_s)(\Psi_{ss} + s\Psi_{sss})\Big]\breve{l}_i\breve{l}_j \\ &+ \Big[\phi_s(\Phi_s - s\Phi_{ss}) - s(\phi - s\phi_s)\Psi_{ss}\Big]\breve{a}_{ij}\Big\} = 0, \end{aligned} \tag{6.7}$$

$$\begin{aligned} \frac{\phi}{2}\Big\{&\Big[s^3\phi_s\Phi_{sss} - 3s\phi_s(\Phi_s - s\Phi_{ss}) - (\phi - s\phi_s)[-3s^2\Psi_{ss} - s^3\Psi_{sss}]\Big]\breve{l}_i\breve{l}_j\breve{l}_k \\ &+ [s\phi_s(\Phi_s - s\Phi_{ss}) - (\phi - s\phi_s)s^2\Psi_{ss}]\breve{l}_k\breve{a}_{ij}(i \to j \to k \to i)\Big\} = 0. \end{aligned} \tag{6.8}$$

Thus, by (6.5)–(6.8), F is a Landsberg metric if and only if the following equations hold:

$$\phi_s \Phi_{sss} + (\phi - s\phi_s)\Psi_{sss} = 0, \tag{6.9}$$

$$\phi_s(\Phi_s - s\Phi_{ss}) - s(\phi - s\phi_s)\Psi_{ss} = 0. \tag{6.10}$$

Proof of the Theorem 1.3. Let $F = \breve{\alpha}\phi(r,s)$ be a Landsberg warped product metric. Then (6.9) and (6.10) hold. Plug (2.3) into (6.9) and (6.10) to obtain

$$3\phi_s\Psi_{ss} + \phi\Psi_{sss} + \phi_s A_{sss} = 0, \tag{6.11}$$

$$\phi_s(\Psi - s\Psi_s) - s\phi\Psi_{ss} + \phi_s(A_s - sA_{ss}) = 0. \tag{6.12}$$

Let $\Omega = \Psi - s\Psi_s$ and $\mu = \phi_s$. Then (6.12) is given as follows:

$$(\phi\Omega)_s + \mu(A_s - sA_{ss}) = 0. \tag{6.13}$$

Differentiating (6.13) with respect to s, we have

$$(\phi\Omega)_{ss} = s\mu A_{sss} - \mu_s(A_s - sA_{ss}). \tag{6.14}$$

Moreover, we have the following:

$$(\phi\Omega)_{ss} = -\phi(\Psi_{ss} + s\Psi_{sss}) - 2s\phi_s\Psi_{ss} + \phi_{ss}\Omega.$$

Plugging (6.11) into above two equations yields

$$\mu_s(A_s - sA_{ss})(\phi - s\phi_s)\Psi_{ss} - \phi_{ss}\Omega = -\frac{\phi - s\phi_s}{s}\Omega_s - \phi_{ss}\Omega. \tag{6.15}$$

Together with (6.13), it yields

$$\mu(\phi_{ss}\Omega + \frac{\phi - s\phi_s}{s}\Omega_s) - \mu_s(\phi_s\Omega + \phi\Omega_s) = 0.$$

Therefore,

$$\frac{\Omega_s}{\Omega}[(\phi - s\phi_s)\mu - s\phi\mu_s] = s\phi_s\mu_s - s\phi_{ss}\mu. \tag{6.16}$$

Hence,

$$(\phi - s\phi_s)\mu - s\phi\mu_s = (\phi - s\phi_s)\phi_s - s\phi\phi_{ss}$$

and

$$s\phi_s\mu_s - s\phi_{ss}\mu = s\phi_s\phi_{ss} - s\phi_{ss}\phi_s = 0.$$

Thus, we conclude that (6.16) holds if and only if

$$(\phi - s\phi_s)\phi_s - s\phi\phi_{ss} = 0, \tag{6.17}$$

or

$$\frac{\Omega_s}{\Omega} = 0 \tag{6.18}$$

satisfies.

The solution of equation (6.17) is given by

$$\phi = \sqrt{\iota_1(r)s^2 + 2\iota_2(r)}, \tag{6.19}$$

where ι_1 and ι_2 are two functions of the variable r. Hence, $F = \breve{\alpha}\phi$ is Riemannian.

Therefore, by (6.18), we have

$$\Psi_{ss} = 0.$$

Its solution is given by

$$\Psi = a_1(r)s + a_2(r), \tag{6.20}$$

where a_1 and a_2 are two functions of r. Combining with (6.12), we have

$$\phi_s[a_2(r) + A_s - sA_{ss}] = 0. \tag{6.21}$$

Note that $\phi_s \neq 0$ (see [10, Proposition 5.1]). Then it follows from (6.21) that

$$A_s - sA_{ss} = -a_2(r).$$

Solving the equation, we get

$$A = \frac{1}{2}s^2 a_3(r) - a_2(r)s + a_4(r). \tag{6.22}$$

If $a_2(r) = 0$, then $A = \frac{1}{2}s^2 a_3(r) + a_4(r)$ and $\Psi = a_1(r)s$, which means that F is Berwaldian by Lemma 1.1.

If $a_2(r) \neq 0$, let $U = \frac{\phi_s}{\phi}$ and $V = \frac{\phi_r}{\phi}$, then

$$\phi_s = U\phi, \quad \phi_r = V\phi.$$

Substituting the above equations into Ψ and A, which are given by (2.4) and (2.5), respectively, we obtain

$$\Psi = \frac{s}{2}V - UA, \quad A = \frac{sV_s + sUV - V}{2(U_s + U^2)}. \tag{6.23}$$

Substituting (6.20) and (6.22) into (6.23), we have

$$U = \frac{sa_1(r) + 2a_2(r)}{s^2 a_1(r) + 2sa_2(r) - 2a_4(r)}, \quad V = \frac{2\Psi + 2UA}{s}.$$

Thus,

$$\begin{aligned} (\ln\phi)_s &= \frac{sa_1(r) + 2a_2(r)}{s^2 a_1(r) + 2sa_2(r) - 2a_4(r)}, \\ (\ln\phi)_r &= \frac{\left(2\,a_1^2(r) + a_1(r)a_3(r)\right)s^2 + 4\,\left(a_1(r) + 1/2\,a_3(r)\right)a_2(r)s - 2a_1(r)a_4(r)}{s^2 a_1(r) + 2a_2(r)s - 2\,a_4(r)}. \end{aligned} \tag{6.24}$$

Integrating the first equation above, we obtain

$$\phi = \exp\left(\int_0^s \frac{ta_1(r) + 2a_2(r)}{t^2 a_1(r) + 2ta_2(r) - 2a_4(r)}\,dt\right) a_0(r),$$

where a_0 is an arbitrary C^∞ function of r. To determine $a_0(r)$, firstly notice that $\phi(r,0) = a_0(r)$. Thus,

$$[\phi(r,0)]_r = a_0'(r).$$

By the second equation of (6.24), it follows that

$$\frac{a_0'(r)}{a_0(r)} = a_1(r).$$

Therefore, the integration with respect to r yields

$$a_0(r) = c\exp\left(\int a_1(r)\,dr\right),$$

where c is the constant of integration. The Theorem 1.3 is proved. □

Acknowledgments. The authors would like to thank the referees for careful reading and helpful comments on this paper.

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Received September 2, 2020, revised December 4, 2020.

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Про метрики Ландсберґа викривленого добутку

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У цій роботі ми обговорюємо клас фінслерових метрик, які називаються фінслеровими метриками викривленого добутку. Ці метрики було вивчено Ченом, Шеном і Жао в 2018. По суті, ми вивчаємо кривину Бервальда фінслерових метрик викривленого добутку. Ми також характеризуємо фінслерові метрики викривленого добутку з ізотропною кривиною Бервальда і встановлюємо, що вони є метриками Рандерса (теорема 1.2). Крім того, для фінслерових метрик викривленого добутку ми розглядаємо важливу проблему фінслерової геометрії про існування фінслерових єдинорогів. Фактично, ми даємо відповідь на питання, чи є метрикою Бервальда метрика Ландсберга, що є фінслеровою метрикою викривленого добутку (теорема 1.3).

Ключові слова: метрика Фінслера викривленого добутку, метрика Ландсберга, ізотропна кривина Бервальда