On the Perturbation of Self-Adjoint Operators with Absolutely Continuous Spectrum

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The perturbation of the linear self-adjoint operator with absolutely continuous spectrum is studied and the inverse problem on finding the perturbation by the given spectrum is solved.

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1. Preliminaries


In the paper, the perturbations of the linear self-adjoint operators in the Hilbert space are studied assuming that the spectrum of the original operator is absolutely continuous. It is shown how the spectral measure corresponding to absolutely continuous spectrum is changed. The inverse theorem on the recovering of the rank-one perturbation by the spectra of the original and perturbed operators is proved.

Let $A$ be a linear self-adjoint operator in a Hilbert space $H$. We consider the perturbation of the operator $A$,

$$B - A = -\varphi^* \sigma \varphi,$$  \hspace{1cm} (1)
where $E$ is a Hilbert space, $\varphi, \sigma$ are bounded operators, $\varphi : H \to E$, $\sigma : E \to E(\sigma = \sigma^*)$. Let us find the resolvent of the operator $B$. Obviously,

$$(B - \lambda I) = (A - \lambda I) - \varphi^*\sigma\varphi.$$ 

Assuming that $\lambda$ does not belong to $\sigma(A) \cup \sigma(B)$ (where $\sigma(A)$ and $\sigma(B)$ are the spectra of the operators $A$ and $B$, respectively), multiply the right-hand side of this equality by $R_\lambda(B) = (B - \lambda)^{-1}$ and its left-hand side by $R_\lambda(A) = (A - \lambda)^{-1}$ to obtain

$$R_\lambda(B) = R_\lambda(A) + R_\lambda(A)\varphi^*\sigma R_\lambda(B). \quad (2)$$

Applying the operator $\varphi$ to both sides of the equality, we get

$$\varphi R_\lambda(A) = (I - \varphi R_\lambda(A)\varphi^*\sigma)\varphi R_\lambda(B).$$

Denote by $N(\lambda)$ the operator function in $E$

$$N(\lambda) = \varphi R_\lambda(A)\varphi^*. \quad (3)$$

We obtain

$$(I - N(\lambda)\sigma)^{-1}\varphi R_\lambda(A) = \varphi R_\lambda(B),$$

where $\lambda$ is such that the operator $(I - N(\lambda)\sigma)^{-1}$ exists and it is bounded.

Substituting $\varphi R_\lambda(B)$ in (2), we get the representation of the resolvent of the operator $B$

$$R_\lambda(B) = R_\lambda(A) + R_\lambda(A)\varphi^*\sigma(I - N(\lambda)\sigma)^{-1}\varphi R_\lambda(A). \quad (4)$$

Thus, we proved the theorem.

**Theorem 1.** Let $A$ be a linear self-adjoint operator in a Hilbert space $H$, and the perturbed operator $B$ have the form of (1). Then the resolvent of the operator $B$ has the form of (4), where $\lambda$ does not belong to $\sigma(A) \cup \sigma(B)$ and is such that the operator $(I - N(\lambda)\sigma)^{-1}$ exists and it is bounded.

Denote by $W(\lambda)$ the operator function in $E$,

$$W(\lambda) = \sigma(I - N(\lambda)\sigma)^{-1}. \quad (5)$$

**Theorem 2** The function $W(\lambda)$ is the Nevanlinna function (the function of class $N$ [2, p. 120]) admitting the representation $W(\lambda) = \sigma(I - N(\lambda)\sigma)^{-1} = \sigma + \int \frac{dw(t)}{t - \lambda}$, where $dw(t)$ is the finite measure.
Proof. Obviously, $N(\lambda)$ is the Nevanlinna function and $\sup \| y N(iy) \| < \infty$ when $y \geq 1$. Thus [3, p. 118],

$$N(\lambda) = \int_{\mathbb{R}} \frac{dF_x}{x - \lambda},$$

(6)

where $F_x = \varphi E_x \varphi^*$, and $E_x$ is the resolution of the identity of the operator $A$.

Let us estimate that the function $W(\lambda)$ is the Nevanlinna function

$$W(\lambda) = \sigma(I - N(\lambda)\sigma)^{-1} - (I - \sigma N^*(\lambda))^{-1} \sigma$$

$$= (I - \sigma N^*(\lambda))^{-1} \left\{ \frac{(I - \sigma N^*(\lambda))\sigma - \sigma(I - N(\lambda)\sigma)}{\lambda - \lambda} \right\} (I - N(\lambda)\sigma)^{-1}$$

$$= (I - \sigma N^*(\lambda))^{-1} \sigma \left\{ \frac{N(\lambda) - N^*(\lambda)}{\lambda - \lambda} \right\} (I - N(\lambda)\sigma)^{-1}$$

$$= W^*(\lambda) \left\{ \frac{N(\lambda) - N^*(\lambda)}{\lambda - \lambda} \right\} W(\lambda).$$

Since $N(\lambda)$ is the Nevanlinna function, then $W(\lambda)$ is also the Nevanlinna function. Besides

$$W(\lambda) - \sigma = \sigma - \sigma(I - N(\lambda)\sigma)^{-1} = (\sigma(I - N(\lambda)\sigma) - \sigma)(I - N(\lambda)\sigma)^{-1}$$

$$= -\sigma N(\lambda)\sigma(I - N(\lambda)\sigma)^{-1}.$$ 

For $y \geq 1$,

$$\sup \| y(W(iy) - \sigma) \| \leq \sup \| \sigma y N(iy)\sigma \| \| (I - N(iy)\sigma)^{-1} \| < \infty.$$ 

Therefore

$$W(\lambda) = \sigma(I - N(\lambda)\sigma)^{-1} = \sigma + \int_{\mathbb{R}} \frac{dw_t}{t - \lambda}.$$ 

(7)

In (2), multiplying by $\varphi^*$ the right-hand side of the equality, and by $\sigma \varphi$ its left-hand side, we obtain

$$\sigma \varphi R_\lambda(B)\varphi^* = \sigma \varphi R_\lambda(A)\varphi^* + \sigma \varphi R_\lambda(A)\varphi^* \sigma \varphi R_\lambda(B)\varphi^*.$$ 

Transforming the expression, we get

$$-I + (I + \sigma \varphi R_\lambda(B)\varphi^*) = \sigma \varphi R_\lambda(A)\varphi^*(I + \sigma \varphi R_\lambda(B)\varphi^*),$$ 

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\[(I + \sigma \varphi R_\lambda(B)\varphi^*)(I - \sigma \varphi R_\lambda(A)\varphi^*) = I,\]

\[\sigma = \left( \sigma + \int \frac{dw_t}{t - \lambda} \right) \left( 1 - \int \frac{dF_t}{t - \lambda} \sigma \right), \quad (8)\]

Thus, the "restriction" of the resolvent of the operator \(B\) on \(E\) by the operator \(\varphi\) can be expressed as the operator function \(W(\lambda)\) (5).

Let us recall the well-known fact [1, p. 95].

**Theorem (M. Riesz).** Suppose the function \(f_t(-\infty < t < \infty)\) is measurable and belongs to \(L^p(-\infty, \infty)\). Then the principal value of the integral

\[\tilde{f}_x = \text{v.p.} \int_{\mathbb{R}} \frac{f_t}{x - t} dt\]

exists and \(\tilde{f}_x \in L^p(-\infty, \infty)\).

2. Direct problem

**Theorem 3.** Suppose \(F_t\) from (6) is a completely continuous operator function and \(\sigma\) is an operator of trace class and it is reversible. Then the function \(w_t\) from (7) is completely continuous, and if it holds \(\int \|F_t'\|^2 dt < \infty, \int \|w_t'\|^2 dt < \infty,\) then \(w_t\) satisfies \(W_t^+F_tW_t^+ - w_t' = 2\pi i w_t'F_tW_t^+,\) where \(W_t^+\) is the limit value of \(W_\lambda\) on the real axis in the semiplane \(\mathbb{C}_+\).

**Proof.** Obviously, the operator function \(w_t\) from (7) is completely continuous because \(F_t\) is a completely continuous operator function and the perturbation is an operator of trace class [2, p. 345].

For \(\forall \lambda \in \mathbb{C} \setminus \mathbb{R}\) from (7) we obtain

\[0 = -\int \frac{d\sigma F_t}{t - \lambda} + \int \frac{dw_t}{t - \lambda} - \int \frac{dw_t}{t - \lambda} \int \frac{dF_s}{s - \lambda} \sigma,\]
Transform the double integral. According to Fubini’s theorem,
\[
\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{dw_t}{t-\lambda} \frac{dF_s}{s-\lambda} = \int_{\mathbb{R}^2} \frac{dw_t dF_s}{(t-\lambda)(s-\lambda)}.
\]

By the conditions of the theorem \( \int_{\mathbb{R}} \|F_t'\|^2 dt < \infty, \int_{\mathbb{R}} \|w_t'\|^2 dt < \infty \), so it holds by Riesz’s theorem
\[
\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{dw_t dF_s}{(t-\lambda)(s-\lambda)} = \int_{\mathbb{R}} \frac{dw_t}{t-\lambda} \Phi_t + \int_{\mathbb{R}} \frac{dF_s}{s-\lambda} \Psi_t,
\]

where
\[
\Phi_t = \int_{\mathbb{R}} \frac{dF_s}{s-t}, \quad \Psi_t = \int_{\mathbb{R}} \frac{dw_t}{t-s}.
\]

The integrals \( \Phi_t \) and \( \Psi_t \) are treated as the principal value of the improper integral and exist by the Riesz theorem.

Hence from (8) we obtain
\[
\int_{\mathbb{R}} \frac{d(\sigma F_t \sigma - w_t)}{t-\lambda} = -\int_{\mathbb{R}} \frac{dw_t}{t-\lambda} \Phi_t - \int_{\mathbb{R}} \frac{dF_s}{s-\lambda} \Psi_t,
\]
\[
\int_{\mathbb{R}} \frac{dt}{t-\lambda} \left[ \sigma F_t \sigma + \int_{0}^{t} w'_s \Phi_s ds + \int_{0}^{t} \Psi_s F'_s ds \right] = \int_{\mathbb{R}} \frac{dw_t}{t-\lambda}.
\]

Thereby, taking into account the properties of the Hilbert transform [3], we have
\[
\sigma F'_t \sigma + w'_t \Phi_t + \Psi_t F'_t \sigma = w'_t
\]
or
\[
(\sigma + \Psi_t) F'_t \sigma = w'_t (I - \Phi_t).
\]

From \( W_\lambda = \sigma (I - N_\lambda \sigma)^{-1} \) it follows that \( W_\lambda (I - N_\lambda \sigma) = \sigma \). Direct \( \lambda \) to the real axis in the semiplane \( \mathbb{C}_+ \). By the conditions of the theorem, there exist \( W^+_x, N^+_x \) which are the limits of \( W_\lambda \) and \( N_\lambda \), respectively. Thus, \( W^+_x (I - N^+_x \sigma) = \sigma \) [5, p. 57].
By the Sokhotski formulas, we get
\[ W^+_x = \pi i w_x + v.p. \int_\mathbb{R} \frac{w'_t}{t-x} dt, \]
\[ N^+_x = \pi i F_x + v.p. \int_\mathbb{R} \frac{F'_t}{t-x} dt. \]

By the definition of \( \Phi_t \) and \( \Psi_t \), we have
\[ \sigma + \Psi_t = W^+_t - \pi i w'_t, \]
\[ I - \Phi_t = I - N^+_t \sigma + \pi i F'_t \sigma. \]

Substituting the obtained expression into (9), we get
\[ (W^+_t - \pi i w'_t)F'_t \sigma = w'_t(I - N^+_t \sigma + \pi i F'_t \sigma). \]

Since \( \sigma \) is reversible, then \( W^+_t \sigma^{-1} = I - N^+_t \sigma \) and we obtain
\[ W^+_t F'_t \sigma - \pi i w'_t F'_t \sigma = w'_t W^+_t \sigma^{-1} + \pi i w'_t F'_t \sigma. \]

Finally we get the expression which connects two measures
\[ W^+_t F'_t - w'_t W^+_t \sigma^{-1} = 2\pi i w'_t F'_t. \] (10)

3. Inverse problem

Suppose \( \dim E = 1 \) and the operator \( \varphi : H \to E \) acts in the following way:
\[ \varphi h = \langle h, g \rangle f, \] where \( f \in E, \| f \| = 1, g \in H, h \in H. \)

The next theorem allows to recover the perturbation by the two measures of the operator functions \( N_\lambda \) and \( W_\lambda \) for the rank-one perturbation.

**Theorem 4.** Let \( A \) be a linear self-adjoint operator with absolutely continuous spectrum and let there be given two completely continuous nondecreasing scalar functions \( F_t, w_t \) such that \( \int |F'_t|^2 dt < \infty, \int |w'_t|^2 dt < \infty. \) If there exists a solution \( W^+_t \) of (10) in the class of the Nevanlinna functions such that \( W^+_t - \pi i w'_t + \int_\mathbb{R} \frac{dw_t}{s-i} = \) const and \( F'_t w'_t \leq 1/\pi^2, \) then there exists a perturbation of the form of (1), where \( \dim E = 1, \varphi h = \langle h, g \rangle f, f \in E, \| f \| = 1, g, h \in H, \) moreover
\[ N(\lambda) = \int_\mathbb{R} \frac{dF_t}{t-\lambda} = \varphi R_\lambda(A)\varphi^*, \]
\[ W(\lambda) = \sigma + \int_{\mathbb{R}} \frac{dw_t}{t-\lambda} = \sigma(I - N(\lambda)\sigma)^{-1}. \]

**Proof.** Let us take \( \forall f : \|f\| = 1 \). Define the function \( N(\lambda) = \int_{\mathbb{R}} \frac{df_t}{t-\lambda} \).

Let \( g \) be the desired vector in the representation for \( \varphi \). Then the following relation holds:
\[ \langle E_t g, g \rangle = F_t, \]
where \( E_t \) is the resolution of the identity of the operator \( A \).

Since the spectrum of the operator \( A \) is simple, then there exists a generating vector \( \xi \in H \) such that
\[ \int_{-\infty}^{x} |g_t|^2 d\rho_t = F_x, \]
where \( \rho_t = \langle E_t \xi, \xi \rangle \) [2, p. 282].

Taking the derivative of the equality, we obtain
\[ |g_x|^2 \rho'_x = F'_x. \]

Therefore for \( \forall x \in \mathbb{R} : \rho'_x \neq 0 \) we have \( |g_x|^2 = \frac{F'_x}{\rho'_x} \), for other \( x' \)s one can assume \( g_x = 0 \).

Taking an arbitrary function \( g_t \), which satisfies the obtained condition, we get
\[ g = \int_{\mathbb{R}} g_t dE_t \xi. \]

Thus the function \( N(\lambda) = \langle R_\lambda(A) g, g \rangle = \varphi R_\lambda(A) \varphi^* \).

By the conditions of the theorem, there exists a solution \( W_t^+ \) of (10),
\[ W_t^+ F'_t - w'_t = 2\pi iw'_t F'_t W_t^+. \]

Solving the equation, we obtain
\[ W_t^+ = \pi iw'_t \pm \sqrt{\frac{w'_t}{F'_t} - \pi^2 w'_t^2}. \]  \( \text{(11)} \)

Since \( F'_t w'_t \leq 1/\pi^2 \), then
\[ \text{Im} W_t^+ = \pi w'_t \geq 0. \]

Define \( \sigma \) by the following expression:
\[ \sigma = W_t^+ - \pi iw'_t + \int_{\mathbb{R}} \frac{dw_s}{s-t}. \]
Note that the defined $\sigma$ is a constant by the conditions of the theorem and $\sigma \in \mathbb{R}$.

Define the function $W(\lambda) = \sigma (I - N(\lambda)\sigma)^{-1}$. By Theorem 2, it holds $W(\lambda) = \sigma + \int_{\mathbb{R}} \frac{d\nu}{\lambda - t}$, where $d\nu_t$ is some finite measure.

From the conditions on the limit value of the function $W(\lambda)$ on real axis (11), by using the Stieltjes–Perron inversion formula [3], we can get that $d\nu_t = dw_t$ and, consequently,

$$W(\lambda) = \sigma + \int_{\mathbb{R}} \frac{dw_t}{t - \lambda} = \sigma (I - N(\lambda)\sigma)^{-1}.$$